# Lesson 36 More on LFT's.

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \qquad L \sim M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$w = L(z) = \frac{az+b}{cz+d} \qquad L_1 \circ L_2 \sim M_1 M_2 \leftarrow \text{matrix multiplication}$$

$$czw + dw = az + b$$

$$z(cw-a) = -dw + b \qquad L^{-1} \sim M^{-1}$$

$$\boxed{z = \frac{-dw+b}{cw-a}} = \frac{dw-b}{-cw+a}$$

$$L(z) = \frac{1\cdot z+0}{0\cdot z+1} \sim M = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= z \leftarrow \text{identity!}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Lemma:** A LFT that fixes 3 pts must be the identity.

Why: $z_1, z_2, z_3$ distinct. $\quad z_j = L(z_j) \quad j = 1, 2, 3$

$$z_j = \frac{az_j+b}{cz_j+d}$$

$$\boxed{cz_j^2 + (d-a)z_j - b = 0}$$

Quadratic with 3 roots! Coeff must all be zero.

$c = 0, \; b = 0$

$d - a = 0 \leftarrow a = d \leftarrow \det \neq 0$ means $a \neq 0, \; d \neq 0$.

$$L(z) = \frac{az+0}{0\cdot z+a} = z \quad \checkmark$$

**Cor:** If $L_1$ and $L_2$ agree at 3 pts, then $L_1 \equiv L_2$.

Why: Apply Lemma to $L_1 \circ L_2^{-1}$.

**Note:** 3 pts in $\mathbb{C}$ determine a circle!

Hmmm. 2 pts determine a line. Secret missing 3rd pt is the pt at $\infty$.

2 pts in $\mathbb{C}$ plus $\infty$ yields a line.

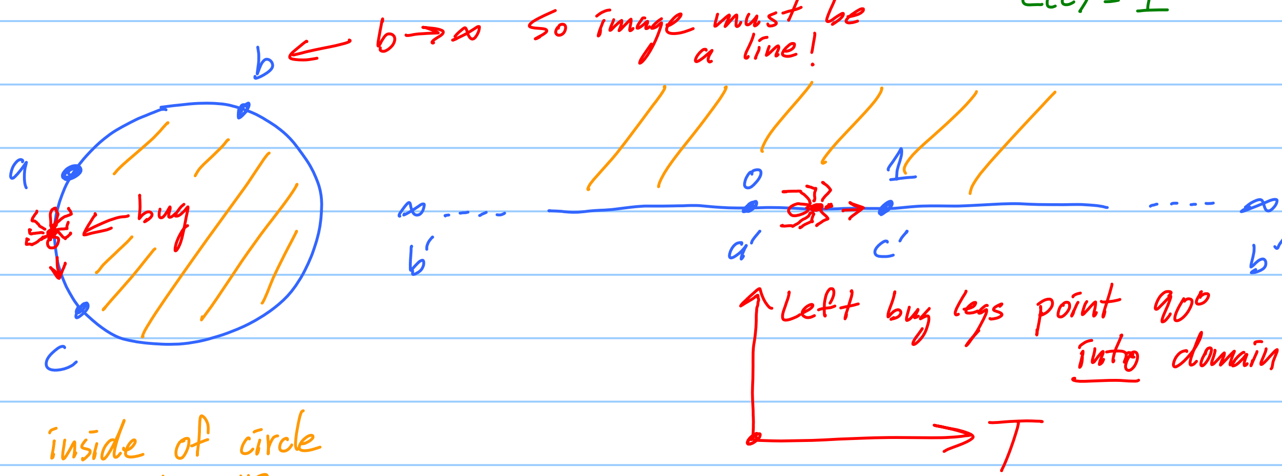
Play around: $L(z) = \dfrac{z-a}{z-b}$ $\quad a \mapsto 0$, $\; b \mapsto \infty$

Next: $a, b, c \in \mathbb{C}$ distinct. Want $L(a)=0$, $L(b)=\infty$, $L(c)=1$

Aha! $\dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}$ works!

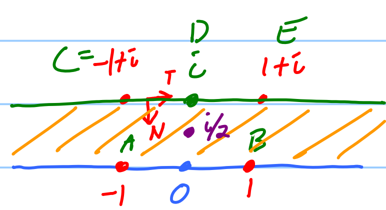
Fact: $L(z) = \dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}$ is the LFT with $L(a)=0$, $L(b)=\infty$, $L(c)=1$

EX:




inside of circle maps 1-1 onto UHP or the LHP

HWK: 2. $L(z) = \dfrac{z-i}{z}$

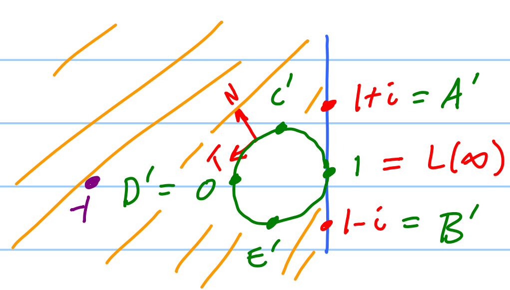



Aha! $L(0)=\infty$. So blue line $\to$ line

$L(1) = \dfrac{1-i}{1} = 1-i$

$L(-1) = \dfrac{-1-i}{-1} = 1+i$

$L(i) = \dfrac{i-i}{i} = 0$



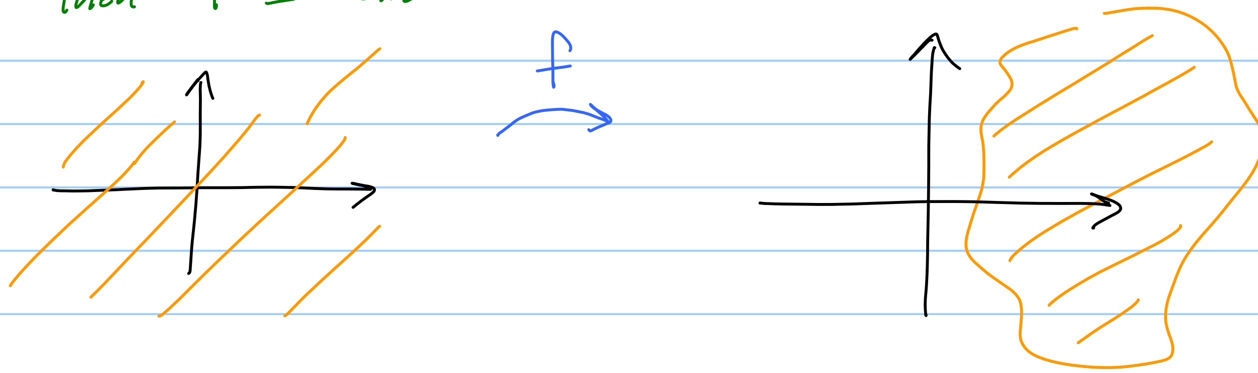
$$L(-1+i) = \frac{-1+i-i}{-1+i} = \frac{1}{1-i} = \frac{1}{2}(1+i)$$

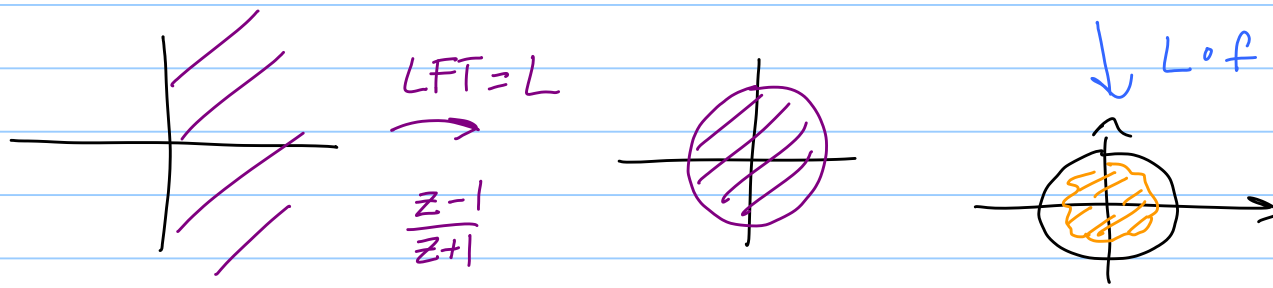
$$L(1+i) = \frac{1+i-i}{1+i} = \frac{1}{1+i} = \frac{1}{2}(1-i)$$

$$L\left(\tfrac{i}{2}\right) = \frac{\frac{i}{2}-i}{\frac{i}{2}} = -1$$

determines region hit by strip

**Old problem:** Suppose $f$ is entire and suppose $\operatorname{Re} f > 0$ on $\mathbb{C}$. Then $f \equiv$ const.



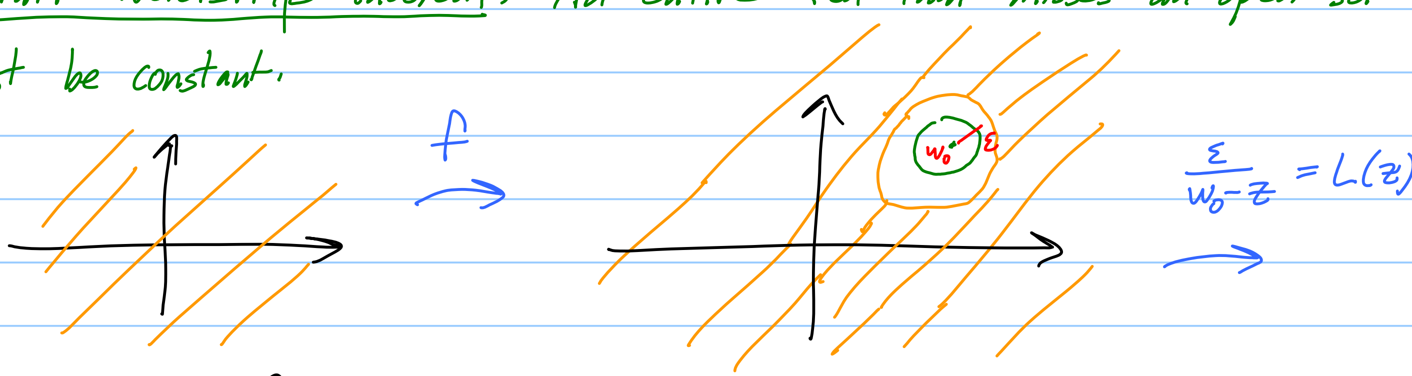



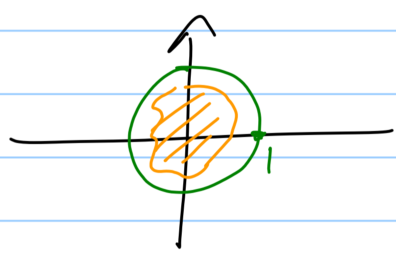

$L \circ f$ is bnd entire fcn! Liouville's

$\Rightarrow L(f(z)) \equiv c$

$f(z) \equiv L^{-1}(c)$ ✓

**Cassorati-Weierstraß Theorem:** An entire fcn that misses an open set must be constant.




$L \circ f$ is bnd entire!

So it is const.

So $f$ is const. ✓