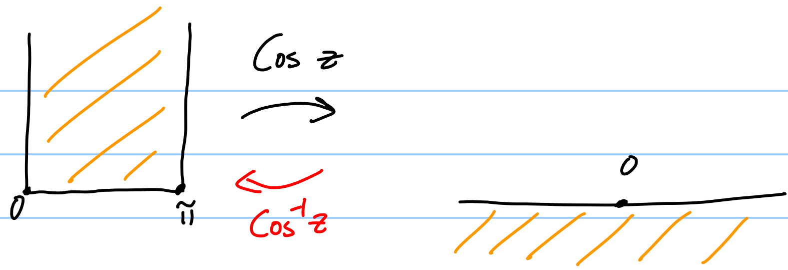
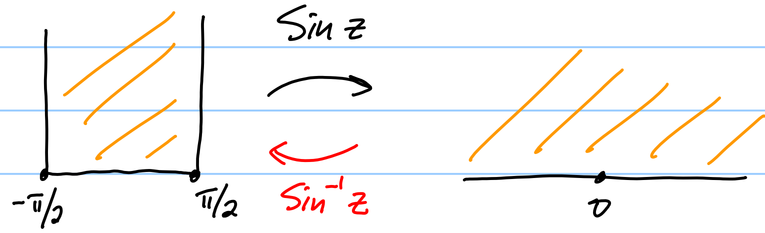
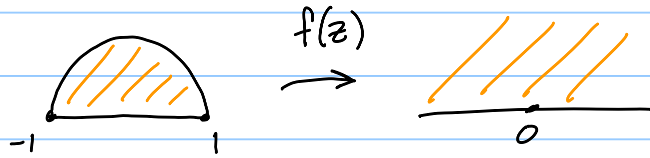
Lesson 40 Schwarz lemma and conformal mappings

38, 39 due Wed.

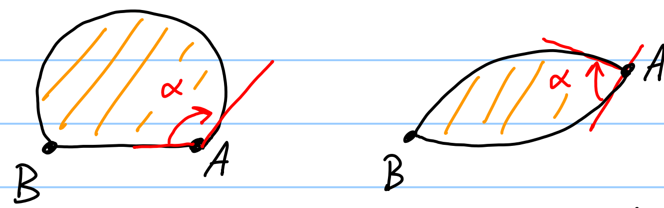




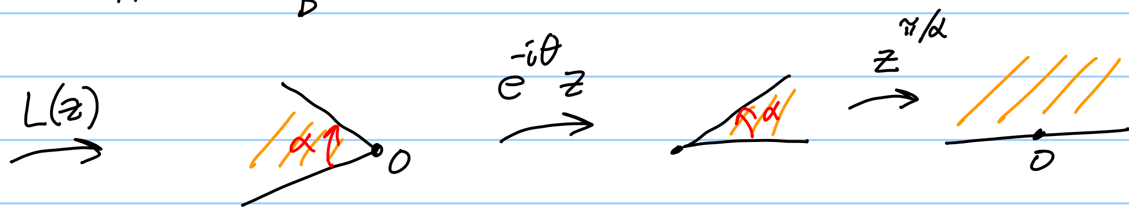
Hwk hints :

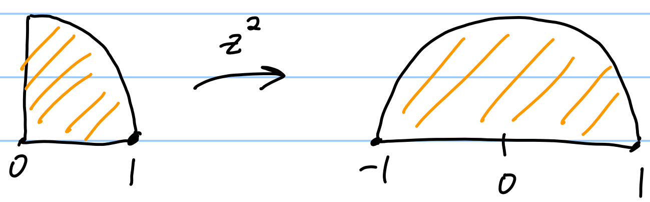


Idea:

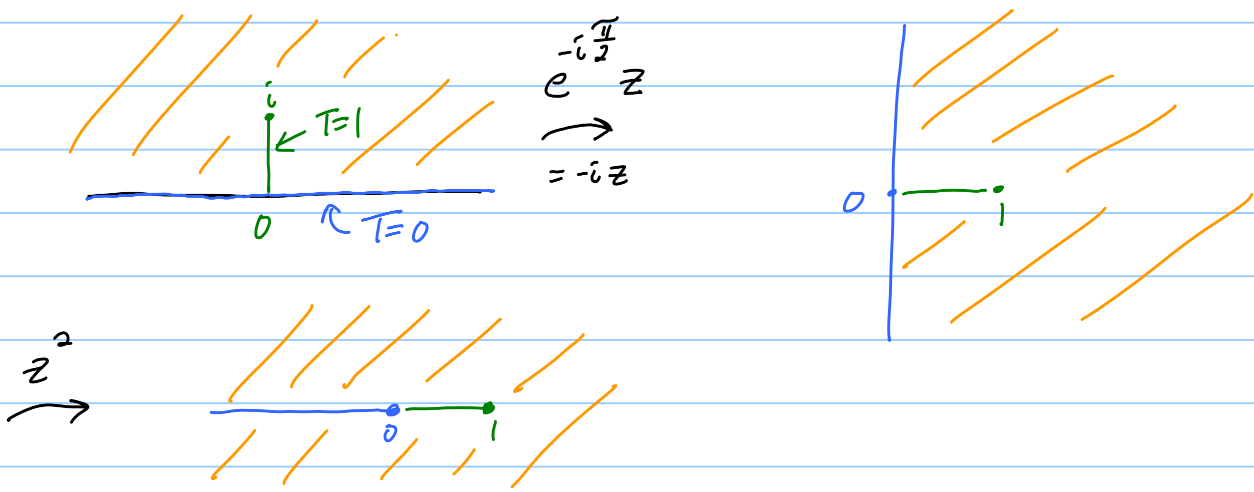


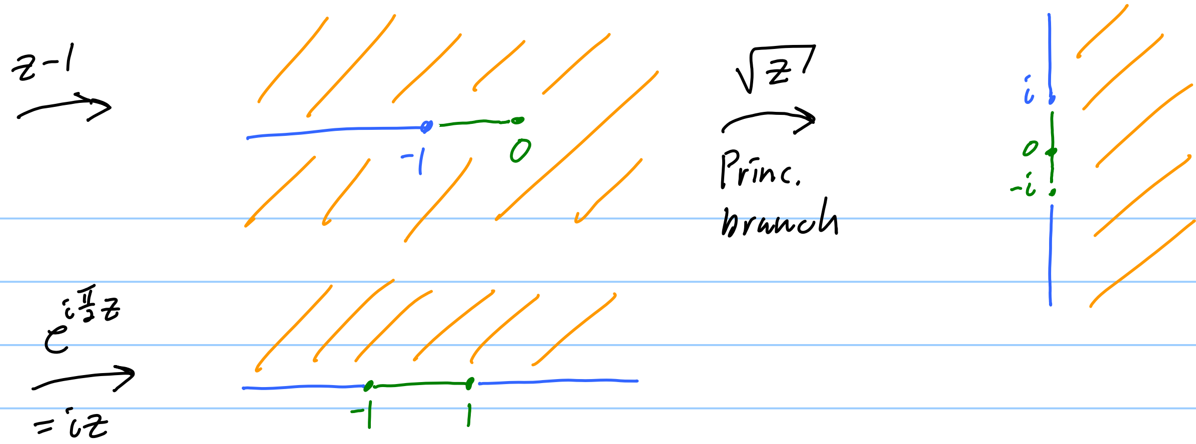
$$L(z) = \frac{z-A}{z-B}$$



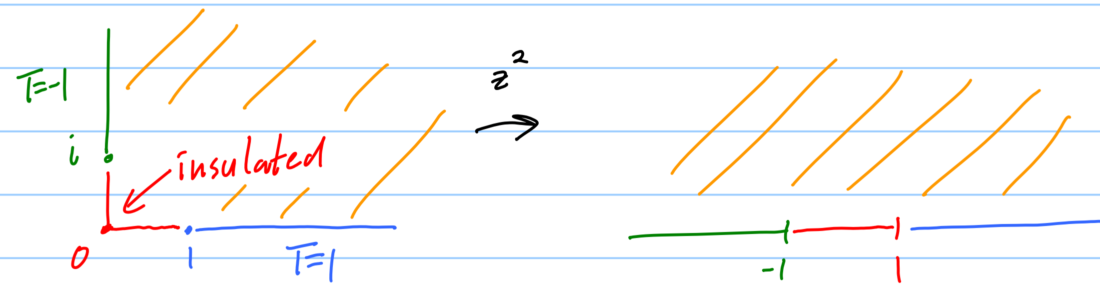


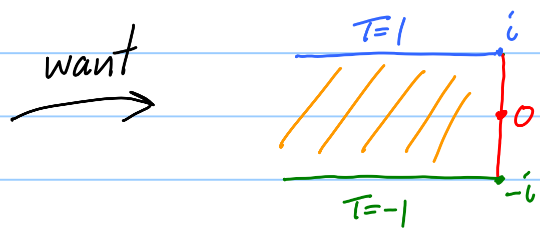
see first hint.





440: 3.





$\text{Sol}^n = \text{Im}\, z = y$

Schwarz lemma: Suppose $f(z)$ is an analytic function on $D_1(0)$ mapping $D_1(0)$ into $D_1(0)$. If $f(0)=0$, then

A) $|f(z)| \le |z|$ for all $z \in D_1(0)$, and

B) $|f'(0)| \le 1$.

Furthermore, if $|f(z)| = |z|$ for some $z \neq 0$

or

if $|f'(0)| = 1$, then

$f(z) = \lambda z$ for some unimodular constant $\lambda = e^{i\theta}$.

Why: Since $f(0)=0$, $f(z) = 0 + a_1 z + a_2 z^2 + \cdots$

$$= z\Big[\underbrace{a_1 + a_2 z + \cdots}_{F(z)}\Big]$$

F analytic on $D_1(0)$. $F(0) = a_1 = \frac{f'(0)}{1!}$.

$$F(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Idea: Maximum princ on $D_r(0)$ where r is close to one.

$f: D_1(0) \to D_1(0)$. So $|f(z)| < 1$ for all $z \in D_1(0)$.

On C_r: $\displaystyle |F(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{\underbrace{|z|}_{=r}} < \frac{1}{r}$

Max Princ: $\displaystyle \max_{\overline{D_r(0)}} |F| = \max_{C_r} |F| < \frac{1}{r}$.

Let $z_0 \in D_1(0)$ and take $|z_0| < r < 1$.

$\displaystyle |F(z_0)| \leq \max_{C_r} |F| < \frac{1}{r} \searrow 1$ as $r \nearrow 1$.

Aha! Let $r \nearrow 1$ to see $|F(z_0)| \leq 1$.

So $\displaystyle \left|\frac{f(z_0)}{z_0}\right| \leq 1$. $|f(z_0)| \leq |z_0|$.

If $z_0 = 0$. Get $|F(0)| = |f'(0)| \leq 1$. ✓

<u>Part 2</u>: If either of the two equalities hold, then

$|F(z_0)| = 1$ at a point in $D_1(0)$. Max Princ $\Rightarrow$

$F \equiv c$, a const. $|F(z_0)| = |c| = 1$. So $c = e^{i\theta}$. ✓

Cor: If $f: D_1(0) \to D_1(0)$ analytic and 1-1, onto, and

<u>$f(0) = 0$</u>, then $f(z) = \lambda z$, $|\lambda| = 1$.

Why: Write $F = f^{-1}$.

$$f(F(z)) = z$$

Chain rule shows $F'(0) = \frac{1}{f'(0)}$.

Schwarz $\Rightarrow$ $|f'(0)| \le 1$. But F satisfies same hyp.

So $|\underbrace{F'(0)}_{\frac{1}{f'(0)}}| \le 1$ too

$$\left.\begin{array}{l} |f'(0)| \le 1 \\ |f'(0)| \ge 1 \end{array}\right\} |f'(0)| = 1.$$

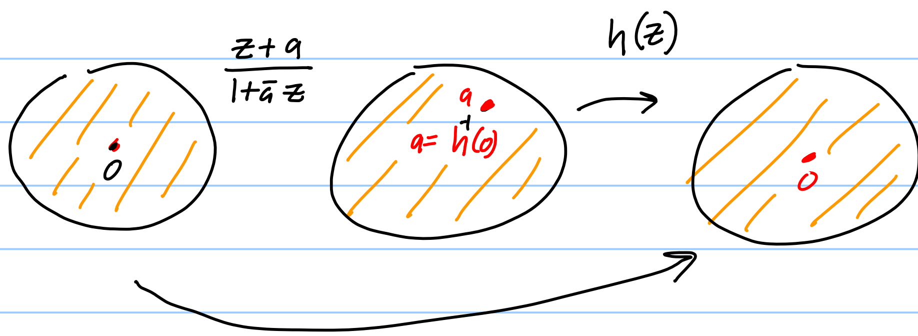
Part 2 of Schwarz $\Rightarrow$ $f(z) = \lambda z$. ✓

Möbius Transformations: $f : D_1(0) \to D_1(0)$ analytic, 1-1, onto.

Fact: $f(z) = e^{i\theta} \dfrac{z-a}{1-\bar{a}z}$ where $a \in D_1(0)$.

Why:

Suppose $h : D_1(0) \to D_1(0)$ 1-1 onto analytic



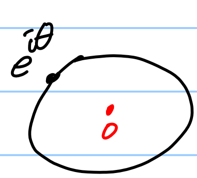
Cor to Schwarz

$\Rightarrow$ $= \lambda z$

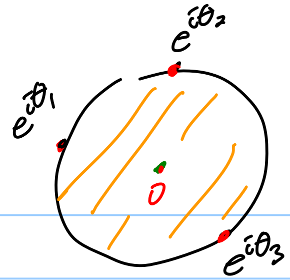
$$h(\underbrace{L(z)}_{w}) = \lambda z = \lambda L^{-1}(w)$$

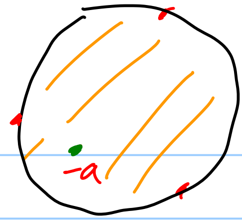
$$h(w) = \lambda L^{-1}(w) = \lambda \frac{z-a}{1-\bar{a}z} \quad ✓$$

Missing step: Show $\underbrace{\dfrac{z-a}{1-\bar{a}z}}_{L(z)}$ map $D_1(0)$ 1-1 onto $D_1(0)$.



$$L(e^{i\theta}) = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} = \frac{e^{i\theta} - a}{e^{i\theta}(e^{-i\theta} - \bar{a})}$$





$$= e^{-i\theta} \cdot \frac{e^{i\theta} - a}{\left(\overline{e^{i\theta} - a}\right)}$$ ← unit modulus