Lesson 2 Complex numbers

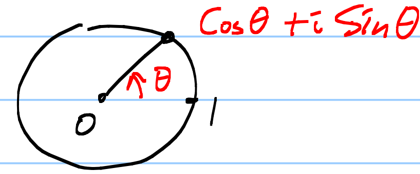
Office hours: T, Th 2:00–3:30 pm in MATH 750 but not this Thurs.

Complex exponential $e^z = e^{x+iy}$

Define: $E(x+iy) = (e^x \cos y) + i\,(e^x \sin y) = e^{x+iy}$ later

Properties: $E(0) = 1$

$E(i\theta) = e^{i\theta} = \cos\theta + i\sin\theta$



$E(z+w) = E(z)E(w)$

$e^{z+w} = e^z e^w$

$E(-z) = 1/E(z) \leftarrow$ Aha! $E(z)$ is never zero!

DeMoivre's Formula: $\left(e^{i\theta}\right)^N = e^{iN\theta}$

Why: $\left(e^{i\theta}\right)^N = \underbrace{e^{i\theta}\cdot e^{i\theta}\cdot \cdots e^{i\theta}}_{N\text{-times}} = e^{i\theta+i\theta\cdots+i\theta} = e^{iN\theta}$

Check: $e^{i(\alpha+\beta)} \overset{?}{=} e^{i\alpha}\cdot e^{i\beta}$

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) \overset{?}{=} \left[\cos\alpha + i\sin\alpha\right]\cdot\left[\cos\beta + i\sin\beta\right]$

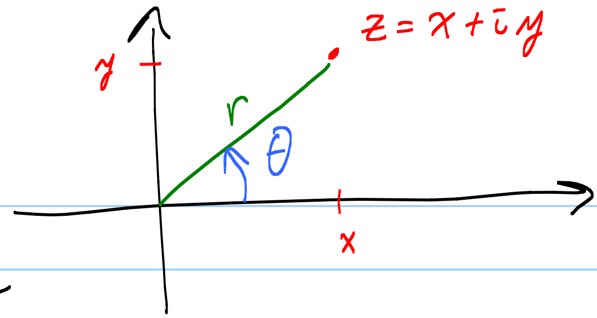
$= \left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\right) + i\left(\cos\alpha\sin\beta + \sin\alpha\cos\beta\right)$ ✓ ✓

Similarly, show $e^{z+w} = e^z\cdot e^w$

Arithmetic on $\mathbb{C}$: Algebra — just like $\mathbb{R}$.

Analysis on $\mathbb{C}$: Topology just like $\mathbb{R}^2$

$\mathbb{C}$



fight for Arg θ here

$x = \operatorname{Re} z$ "real part of z"

$y = \operatorname{Im} z$ "imaginary part"

$r = |z|$ "modulus of z"

$\theta = \underline{\operatorname{Arg}} \, z$ ← Principal argument of z

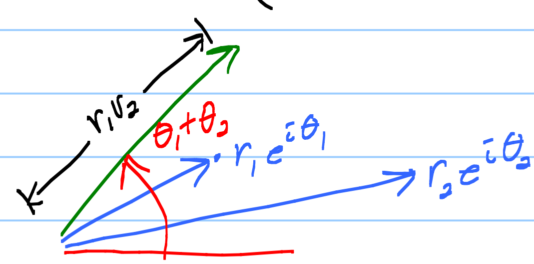
$-\pi < \theta \leq \pi$

$\arg z = \{\theta + 2n\pi : n \in \mathbb{Z}, \ \theta = \operatorname{Arg} z\}$ ← set

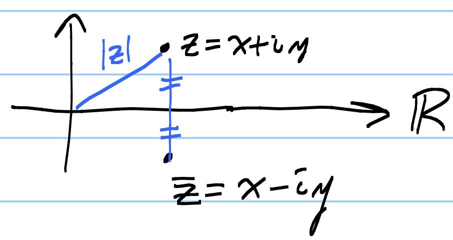
Polar form: $z = x + iy = r\cos\theta + i r \sin\theta = \underline{re^{i\theta}}$

EX: $z^n = (re^{i\theta})^n \underset{\text{assoc law}}{=} r^n (e^{i\theta})^n \underset{\text{DeM}}{=} r^n e^{in\theta}$

EX: $z_1 \cdot z_2 = (r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}$



Easy facts:



$|z| = \sqrt{x^2 + y^2}$

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

$\overline{\left(z/w\right)} = \dfrac{\bar{z}}{\bar{w}}$ if $w \neq 0$

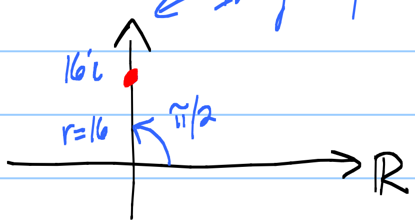
$|\bar{z}| = |z|$

$|zw| = |z| \cdot |w| \leftarrow \left| r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \right| = r_1 r_2 \underbrace{\left| e^{i(\theta_1 + \theta_2)} \right|}_{\text{on unit circle. So } 1 \cdot 1 = 1.}$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \overline{e^{i\theta}}$$

Finding complex roots: $z^4 = 16i$



$(re^{i\theta})^4 = 16i$ ← want

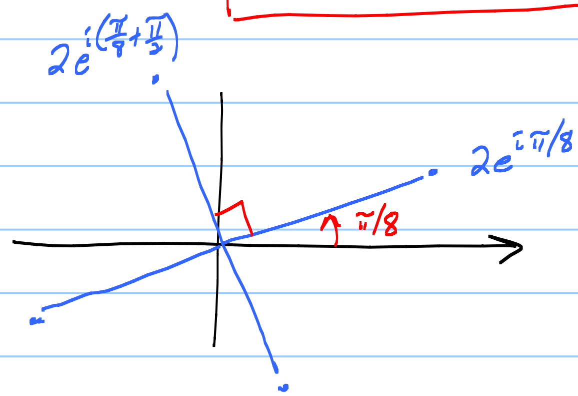
$r^4 e^{i4\theta} = 16 \cdot e^{i\pi/2}$

Need $\boxed{r^4 = 16}$ $\qquad 4\theta = \frac{\pi}{2} + 2n\pi \quad n = 0, \pm 1, \pm 2, \ldots$

$\boxed{r = 2}$ $\qquad \boxed{\theta = \frac{\pi}{8} + n\frac{\pi}{2}} \quad n \in \mathbb{Z}.$

∞-many?

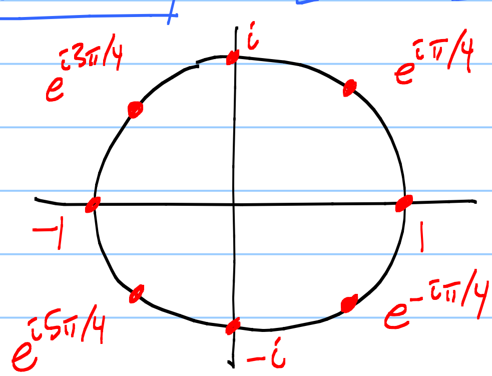
$z = 2e^{i\left(\frac{\pi}{8} + n\frac{\pi}{2}\right)}$



Only 4 4-th roots of $16i$.

N-th roots of unity: $z^N = 1$

8-th roots of unity



Topology of $\mathbb{C}$: Distance function in $\mathbb{C}$:

$$\mathrm{dist}(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|$$

same as $\mathbb{R}^2$ by definition.

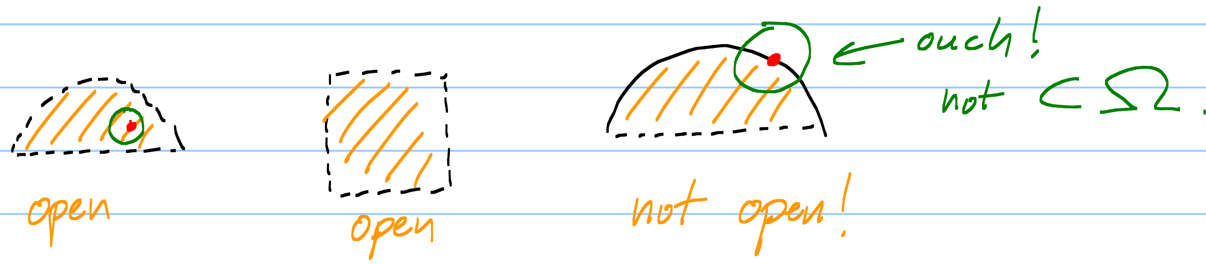
$$D_R(z_0) = \{ z : |z - z_0| < R \}$$

"disc of radius r about z_0"

<u>Def^n</u>: $\Omega \subset \mathbb{C}$ is called <u>open</u> if, for any $z_0 \in \Omega$, there is a disc $D_R(z_0) \subset \Omega$ (with $R > 0$).

<u>Note</u>: R usually depends on the z_0.

<u>EX</u>: Ω



Sierpinski snowflake: