Lesson 4 Analytic functions

Read Chap 2.

Lessons 1,2,3 due Wed at beginning of class. Stapled together.

Geometric series: $\underbrace{1+z+\cdots+z^N}_{S_N} = \frac{1}{1-z} - \frac{z^{N+1}}{1-z} = \frac{1-z^{N+1}}{1-z}$

$$S_N = 1 + z + \cdots + z^N$$
$$-\quad zS_N = z + z^2 + \cdots + z^N + z^{N+1}$$
$$(1-z)S_N = 1 - z^{N+1}$$

$$(e^{i\theta})^N = (\cos\theta + i\sin\theta)^N = e^{iN\theta}$$

Fourier series fact: $\underbrace{\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos N\theta}_{C_N} = \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$

$$C_N = \mathrm{Re}\left[1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{iN\theta}\right] - \frac{1}{2}$$

$$= \mathrm{Re}\left[\underbrace{1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^N}_{\frac{1-(e^{i\theta})^{N+1}}{1-e^{i\theta}}}\right] - \frac{1}{2}$$

$$= \text{clever algebra} \ldots = \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \;!$$

The "point at infinity"

$-\infty \cdots \longleftarrow \; 0 \; \mathbb{R} \longrightarrow \cdots +\infty$

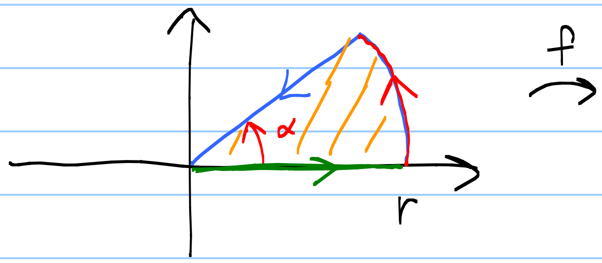
EX: $f(z) = \frac{1}{z}$ $\qquad f(re^{i\theta}) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

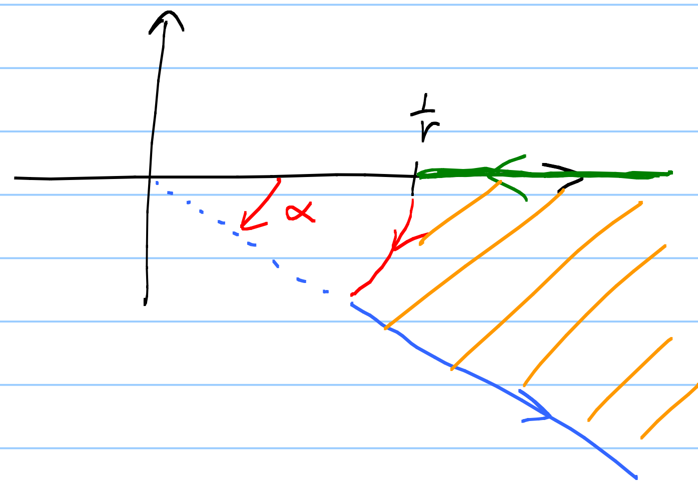
$$\lim_{r\to 0} |f(re^{i\theta})| = \lim_{r\to 0} \frac{1}{r} = +\infty$$

Say $\lim_{z\to 0} f(z) = \infty$, meaning

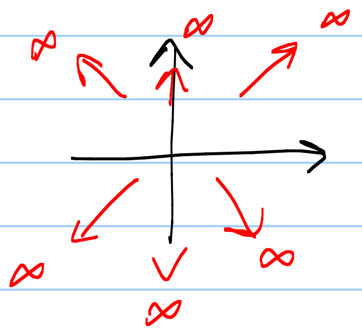
Given any $M > 0$, no matter how big, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z| < \delta$ and $z \neq 0$.

Picture:





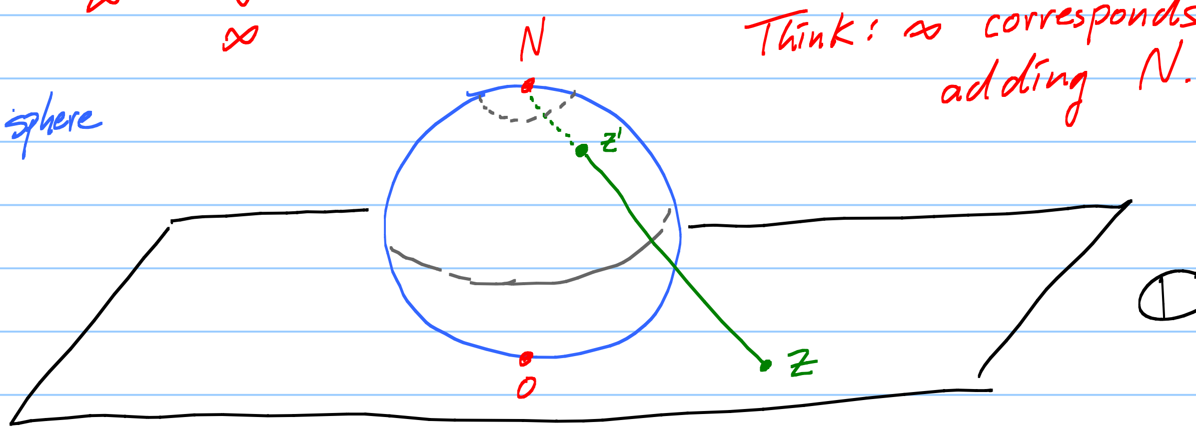
$$f(re^{i\theta}) = \frac{1}{r}e^{-i\theta}$$



$\mathbb{C}$

Think: ∞ corresponds to adding N.

Riemann sphere



$[-\infty, \infty]$ "compactifies" $\mathbb{R}$

"Disc about ∞" $= D_\varepsilon(\infty) = \{z : |z| > \frac{1}{\varepsilon}\}$ ← "polar cap"

Ex: $f(z) = \frac{1}{(z-a)^2}$ $\quad |f(z)| = \frac{1}{|z-a|^2}$

$\left[\lim_{z \to a} |f(z)| = \lim_{z \to a} \frac{1}{|z-a|^2} = +\infty\right]$ ← Meaning of $\lim_{z \to a} f(z) = \infty$

from $\hat{\mathbb{C}}$

Extended complex plane: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Def^{n}s: f complex diff'ble at a means

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists.}$$

Def^n: f is analytic (holomorphic) on an open set Ω if f is complex diff'ble at every pt in Ω.

Analytic fcns satisfy all the rules of freshman calc.

Ex: $[fg]' = f'g + fg'$

$$DQ = \frac{f(z)g(z) - f(a)g(a)}{z-a} = \frac{\big[(f(z) - f(a)) + f(a)\big]g(z) - f(a)g(a)}{z-a}$$

$$= \frac{f(z)-f(a)}{z-a} \cdot g(z) + f(a)\frac{g(z)-g(a)}{z-a}$$

$$\downarrow \qquad \downarrow \qquad\qquad \downarrow \qquad \downarrow$$

$$f'(a) \cdot g(a) + f(a) \cdot g'(a) \quad ✓$$

oops! Need $\lim_{z \to a} g(z) = g(a)$

Fact: g complex diff'ble at $a \implies g$ continuous at a.

Why: $DQ = \dfrac{g(z) - g(a)}{z - a} = g'(a) + E(z)$ where $E(z) \to 0$ as $z \to a$

$$g(z) = g(a) + g'(a)(z-a) + E(z)(z-a)$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$ as $z \to a$

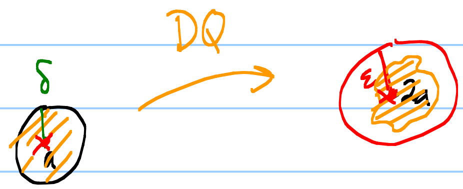
$$g(a) + g'(a) \cdot 0 + 0 \cdot 0 = g(a) \quad ✓$$

Cauchy-Riemann Equations (C-R Eqns)

EX: $f(z) = z^2$. $\quad \dfrac{z^2 - a^2}{z - a} = z + a \to 2a$ as $z \to a$

$f'(a) = 2a$

$f'(z) = 2z$

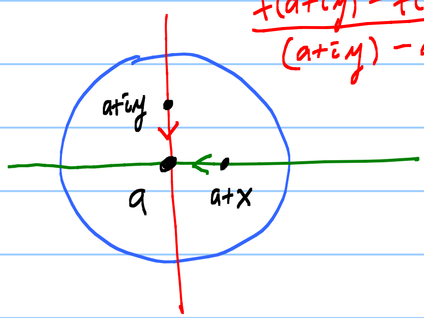


EX: $f(z) = \bar{z}$ $\quad z = a + re^{i\theta}$ $\quad DQ = e^{-2i\theta} \leftarrow$ Ouch! Not a single $f'(a) \in \mathbb{C}$

$f(x+iy) = x - iy$

Idea: If $f(x+iy) = u(x,y) + iv(x,y)$ is complex diff'ble

$\dfrac{f(a+iy) - f(a)}{(a+iy) - a}$



$\dfrac{f(a+x) - f(a)}{(a+x) - a}$

better be the same limit as $x \to 0$, $y \to 0$

Need C-R Eqns: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$