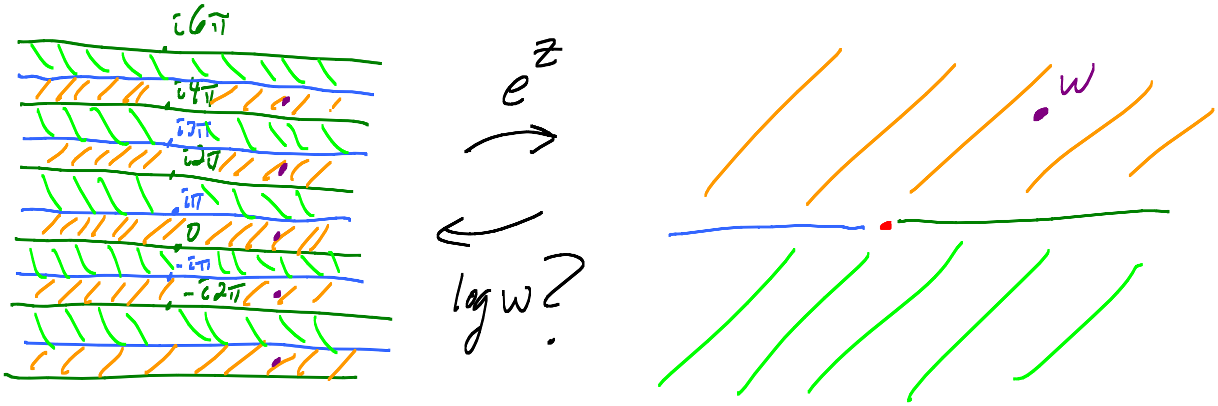
Lesson 9 Complex log functions



Think: $w = e^z$. Then $z = \log w$

$$w = e^{x+iy} = e^x e^{iy}$$

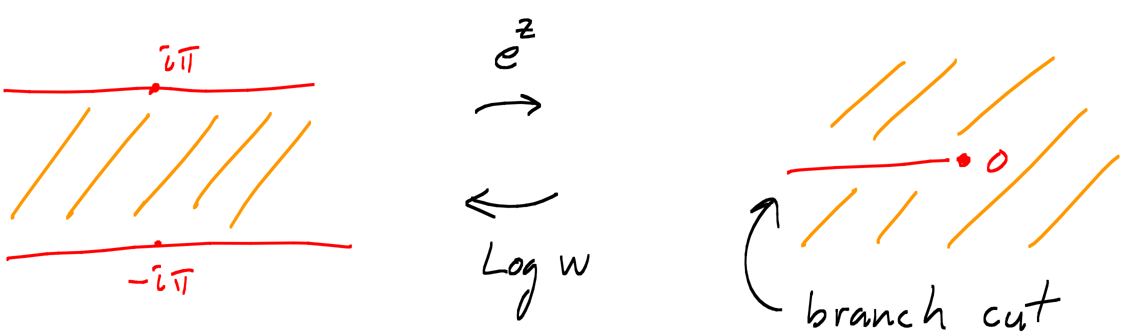
Aha! $e^x = |w|$. So $\boxed{x = \operatorname{Ln}|w|}$

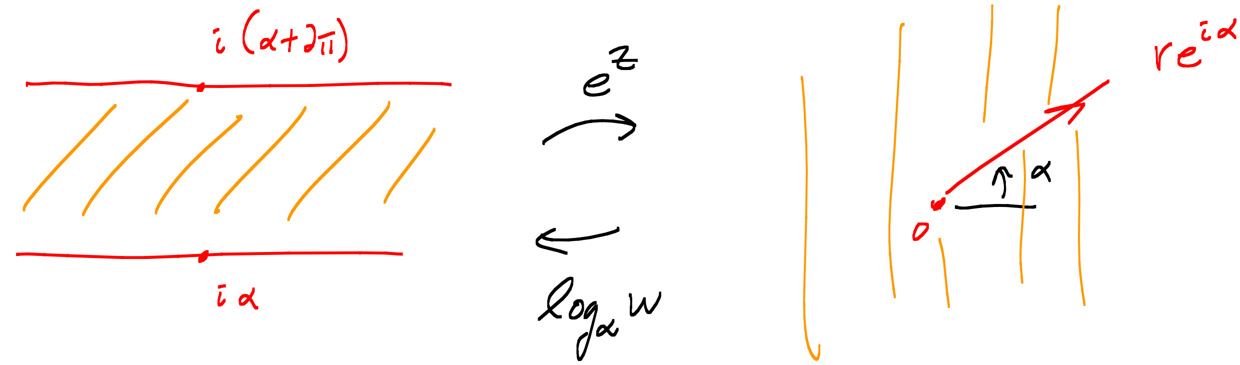
$y \in \arg w$ $\quad y = \operatorname{Arg} w + 2n\pi i$ ← all possible y's

My notation: $\underset{\text{a set}}{\underline{\log w}} = \{ z : e^z = w \} = \{ \operatorname{Ln}|w| + i\theta : \theta \in \arg w \}$

$\operatorname{Log} w = \operatorname{Ln}|w| + i \operatorname{Arg} w$ ← Principal branch

$\log_\alpha w = \operatorname{Ln}|w| + i\theta$ where $\theta \in \arg w$ with $\alpha < \theta < \alpha + 2\pi$



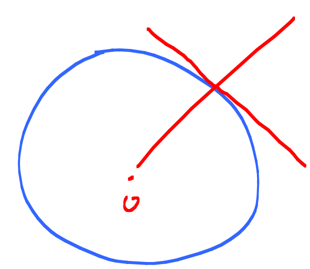


Popular branch :

Fact: $\operatorname{Log} w$, $\log_\alpha w$ are analytic on orange stuff.

Polar C-R Eqns : $f(re^{i\theta}) = U(r,\theta) + i V(r,\theta)$

$$\begin{cases} U_r = \frac{1}{r} V_\theta \\ V_r = -\frac{1}{r} U_\theta \end{cases}$$



Why: $f(x+iy) = u(x,y) + i v(x,y)$

$$\begin{cases} U(r,\theta) = u(r\cos\theta, r\sin\theta) \\ V(r,\theta) = v(r\cos\theta, r\sin\theta) \end{cases}$$

Chain rule : $U_r = u_x \cos\theta + u_y \sin\theta$

$U_\theta = u_x \cdot (-r\sin\theta) + u_y \cdot (r\cos\theta)$

$$\frac{1}{r}U_\theta = -u_x \sin\theta + u_y \cos\theta$$

Similarly for V.

$$\begin{bmatrix} U_r & V_r \\ \frac{1}{r}U_\theta & \frac{1}{r}V_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \overset{!}{\Longleftarrow} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Same idea shows $\Longrightarrow$ too!

<u>EX</u>: $\operatorname{Log} w = \operatorname{Log} re^{i\theta} = \underbrace{\operatorname{Ln} r}_{U(r,\theta)} + i\underbrace{\theta}_{V(r,\theta)}$ where $\theta = \operatorname{Arg} w$.

Polar C-R Eqns ✓

My notation: $\operatorname{Ln} x$ = <u>real</u> natural log

What is $\frac{d}{dz}(\log z)$?

Always true: $e^{\log z} = z$

<u>Not</u> always true: $\log e^z = z + 2\pi n i$ for some $n \in \mathbb{Z}$.

Ouch! $\log z_1 z_2 = \log z_1 + \log z_2 + 2\pi n i$ for some $n \in \mathbb{Z}$

But: $\mathrm{Log}\, z_1 z_2 = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$ if z_1, z_2 in first quadrant

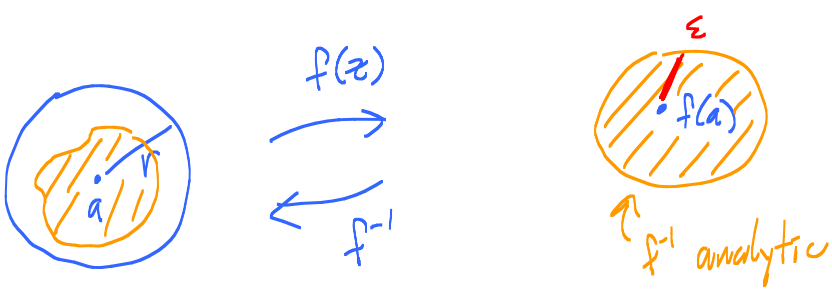
Calculation: True: $e^{\log z} = z$

$$\frac{d}{dz}\left[E(\log z) = z \right]$$

$$\underbrace{\underbrace{E'(\log z)}_{E(\log z)}}_{z} \frac{d}{dz} \log z = 1$$

Aha! $\boxed{\frac{d}{dz}(\log z) = \frac{1}{z}}$

Inverses of analytic fcns are analytic: Inverse function theorem: If f is analytic on $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is analytic near $f(a)$.



f: 1-1 onto orange open sets

Why: $f(x+iy) = u(x,y) + i v(x,y)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Jacobian matrix: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$\mathbb{R}^2 \to \mathbb{R}^2$ inverse fcn thm:

Jacobian of inverse = $(\text{Jacobian})^{-1}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Aha! preserves C-R Eqns.

C-R eqns for $f \implies$ C-R Eqns for inverse!

Fantastic fact: All trig, inverse trig, hyp trig, inverse hyp trig can be written in terms of e^z and $\log z$!

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

$(+)$: $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$(-)$: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\tan z = \frac{\sin z}{\cos z}$$

etc.

$$\cos^{-1} w = z \quad ?$$

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\left[2w = e^{iz} + e^{-iz} \right] \cdot e^{iz}$$

$$\underbrace{e^{i2z}}_{(e^{iz})^2} + 1 = 2we^{iz}$$

$$(e^{iz})^2 - 2w(e^{iz}) + 1 = 0$$ quad equ!

$$e^{iz} = \frac{2w + \sqrt{4w^2 - 4}}{2}$$

$$z = \frac{1}{i} \log\left(w + \sqrt{w^2 - 1}\right)$$