Lesson 21 Power Series

19, 20, 21 due Wed

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad R = \infty$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad R = \infty$$

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad R = \infty$$

Two meanings of "complete":

1) $\mathbb{C}$ is a complete field: Every poly with complex coeffs has a root in $\mathbb{C}$

2) $\mathbb{C}$ is a complete metric space: Every Cauchy sequence of complex #s has a limit in $\mathbb{C}$.

Def^n: A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there is an N such $|a_n - a_m| < \varepsilon$ when $n, m > N$.

Remark: $\mathbb{R}, \mathbb{Q}$ are fields, not complete fields.

$\mathbb{R}$ is a complete metric space. $\mathbb{Q}$ is not.

Lemma: Comparison test: $M_j > 0$. Suppose $\sum_{j=1}^{\infty} M_j < \infty$.

If $z_j \in \mathbb{C}$ and $|z_j| < M_j$, then $\sum_{j=1}^{\infty} z_j$ converges.

Why: Let $S_N = \sum_{1}^{N} z_j$.

$M > N$: $$|S_M - S_N| = \left|\sum_{j=N+1}^{M} z_j\right| \le \sum_{j=N+1}^{M} |z_j| < \underbrace{\sum_{j=N+1}^{M} M_j}_{\text{Tail end of a conv. series. Can make } < \varepsilon}$$

See S_N is a Cauchy seq. So series converges.

for big enough N.

Remark: Taking $M_j = |z_j|$ shows that absolute conv. $\Rightarrow$ convergence!

Ratio test: Given a series $\sum_{n=0}^{\infty} z_n$, $z_n \in \mathbb{C}$. Suppose that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$ exists. Call it L.

If $L<1$, the series converges.

If $L>1$, it diverges.

If $L=1$, the test fails $\leftarrow$ $\sum_1^{\infty} \frac{1}{n}$ diverges, but $\sum_1^{\infty} \frac{1}{n^2}$ converges.

Why: If $L<1$: Compare tail end of series to a convergent geometric series. Comparison test $\Rightarrow$ conv.

$$\left|\frac{a_{n+1}}{a_n}\right| \to L < r < 1$$

So $\exists N$ such that $\left|\frac{a_{n+1}}{a_n}\right| < r$ for $n > N$.

$$|a_m| < r|a_{m-1}| < r\cdot r|a_{m-2}| < \cdots < r^{m-N}|a_N|$$

$$\sum_{m=N+1}^{\infty} |a_m| < \sum_{m=N+1}^{\infty} |a_N| r^{m-N} = \frac{|a_N|}{r^N} \underbrace{\sum_1^{\infty} r^m}_{\frac{1}{1-r}} \checkmark$$

If $L>1$. $\left|\frac{a_{n+1}}{a_n}\right| \to L > r > 1$.

So $\left|\frac{a_{n+1}}{a_n}\right| > r$ for $n > N$.

$$|a_{n+1}| > r\,|a_n| > |a_n|.$$

Terms <u>strictly</u> grow in modulus starting after N.

terms $\not\to 0$. Series must diverge.

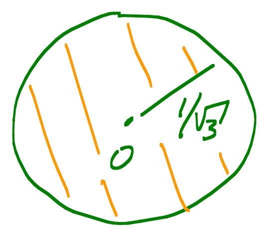
EX: Find radius of conv of $\displaystyle\sum_{n=0}^{\infty} \underbrace{n^2 3^n z^{2n}}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2\, 3^{n+1}\, z^{2(n+1)}}{n^2\, 3^n\, z^{2n}}\right| = \left(\frac{n+1}{n}\right)^2 \cdot 3\left|z^2\right| = \left[1+\frac{1}{n}\right]^2 \cdot 3|z|^2$$

$$\to 3|z|^2 \begin{array}{ll} < 1 & \text{conv.} \\ > 1 & \text{div.} \end{array}$$

$|z| < \frac{1}{\sqrt{3}}$ Conv.

$|z| > \frac{1}{\sqrt{3}}$ div.



$R = \frac{1}{\sqrt{3}}$

EX: $\displaystyle e^z = \sum_0^{\infty} \frac{z^n}{n!}$

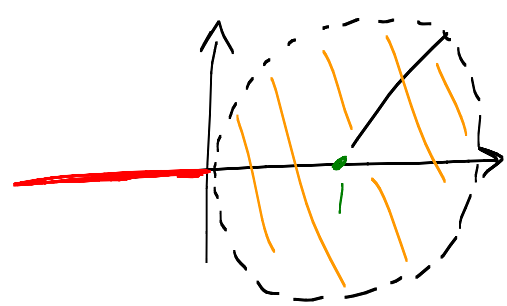
$$\left|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}\right| = \frac{|z|}{n+1} \to 0 < 1$$

Converges for all z!

$R = \infty$.

EX: $\displaystyle\sum_{n=0}^{\infty} z^{n!}$? $R=1$.

EX: $\displaystyle f(z) = \operatorname{Log} z = \sum_{n=0}^{\infty} a_n (z-1)^n$



$R \geq 1$

$f(x) = \ln x \to -\infty$ as $x \searrow 0$. So $R \leq 1$ too. So $R = 1$.

$f(z) = \text{Log } z \qquad a_0 = \frac{f(1)}{0!} = \frac{\text{Log } 1}{1} = \frac{\text{Ln } 1 + i \cdot 0}{1} = 0$

$f'(z) = \frac{1}{z} \qquad a_1 = \frac{f'(1)}{1!} = \frac{1/1}{1} = 1$

$f'' = \frac{-1}{z^2} \qquad a_2 = \frac{f''(1)}{2!} = \frac{-1/1^2}{1 \cdot 2} = -\frac{1}{2}$

$f''' = \frac{2}{z^3} \qquad a_3 = \frac{2/1^3}{3!} = \frac{1}{3}$

$\vdots \qquad \vdots$

Math induction $\quad a_n = \frac{(-1)^{n+1}}{n}$

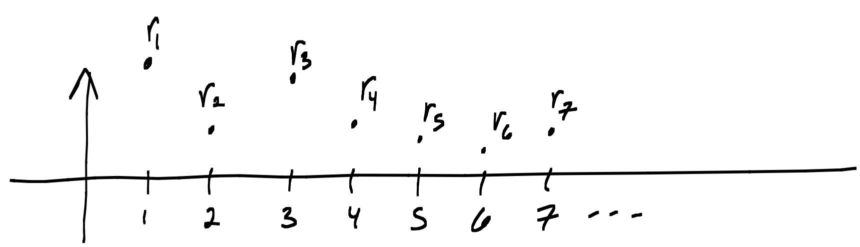
$$\text{Log } z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots$$

Famous Formula: Hadamard's Formula : $\sum_{n=0}^{\infty} a_n z^n$. R. of C. $= R$

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

EX: $\sum_{n=0}^{\infty} z^{n!} \qquad |a_n| = \begin{cases} 0 & n \text{ not a factorial} \\ 1 & n = m! \text{ for some } m. \end{cases}$

Def^n:



$\text{Sup}\{r_n : n \geq N\}$ = "Supremum"

can only go down or stay same as N grows.

EX: $\sup\{r : r \in \mathbb{Q},\ r^2 < 2\} = \sqrt{2}$ ← least upper bound.

$\sup\{r_n : n \geq N\}$ is non-increasing and bounded below.

It must converge!

Def^n: $\limsup_{n\to\infty} r_n = \lim_{N\to\infty} \sup\{r_n : n \geq N\}$.