Lesson 30 Argument Principle & Rouché's Thm

28, 29, 30 due Wed.

$$\operatorname{Res}_{z_0} \frac{f'(z)}{f(z)} = \left(\begin{array}{c}\text{order of the zero}\\ \text{of } f \text{ at } z_0\end{array}\right) \quad \text{or} \quad -\left(\begin{array}{c}\text{order of the}\\ \text{pole of } f \text{ at } z_0\end{array}\right)$$

Why: $f(z) = a_N (z-z_0)^N + \cdots = (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z)}\Big]$ $\leftarrow F(z_0) = a_N \neq 0$

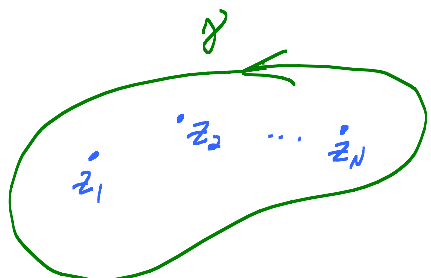
$$f'(z) = N(z-z_0)^{N-1} F(z) + (z-z_0)^N F'(z)$$

So $$\frac{f'(z)}{f(z)} = \underbrace{\frac{N}{z-z_0}}_{\text{Princ. Part}} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } z_0}$$

N ← Res!

Observe: Works at a pole with $-N$ in place of N!

Argument Principle:



f analytic inside and on a simple closed curve γ, non-vanishing on γ, with (finitely) many zeroes of mult. m_k at z_k, $k = 1, \ldots, N$.

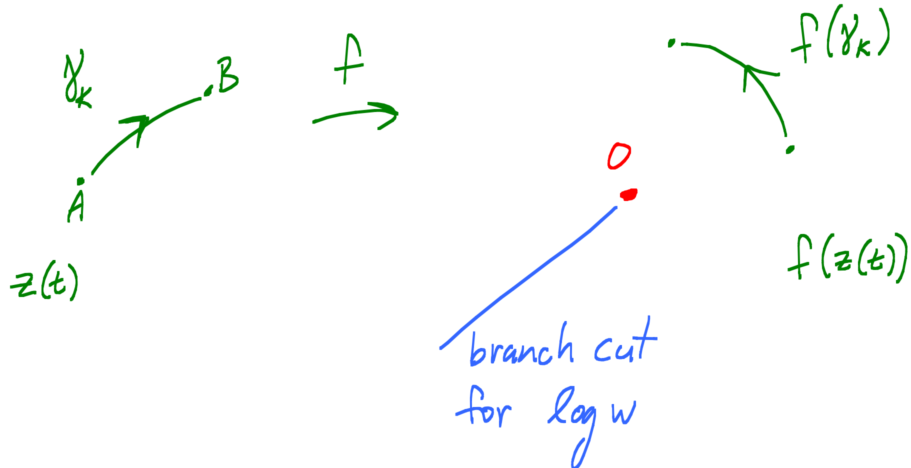
Then $$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\text{Total number of zeroes}\\ \text{of } f \text{ inside } \gamma\end{array}\right) = \sum_{k=1}^{N} m_k$$

counted with multiplicity

Remark: Can allow poles inside too. Get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\text{Total}\\ \#\\ \text{zeroes}\end{array}\right) - \left(\begin{array}{c}\text{Total}\\ \#\\ \text{poles}\end{array}\right)$$ ← counted with mult.

Why "Argument" princ? $\frac{f'(z)}{f(z)} \overset{?}{=} \frac{d}{dz}(\log f)$ yes.



Now $\int_{\gamma_k} \frac{f'}{f}\,dz = \int_{\gamma_k} \frac{d}{dz}(\log f)\,dz = \log f \Big|_A^B$

$= \left(\text{Ln}|f(B)| + i \arg f(B)\right) - \left(\text{Ln}|f(A)| + i \arg f(A)\right)$

$= \underbrace{\left(\text{Ln}|f(B)| - \text{Ln}|f(A)|\right)} + i\, \Delta \arg f \Big|_A^B$

Aha! Add up γ_k's
going around γ,
Get pairwise cancellation.
First last cancel too.

Improved version: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{matrix}\text{Total}\\ \#\\ \text{zeroes}\end{matrix}\right) = \frac{1}{2\pi i}\, i\, \Delta_\gamma \arg f(z)$

$= \left(\begin{matrix}\#\text{ times } f(z) \text{ spins}\\ \text{around the origin}\end{matrix}\right)$

EX:

Spin around <u>twice</u>.
f has two zeroes inside γ.

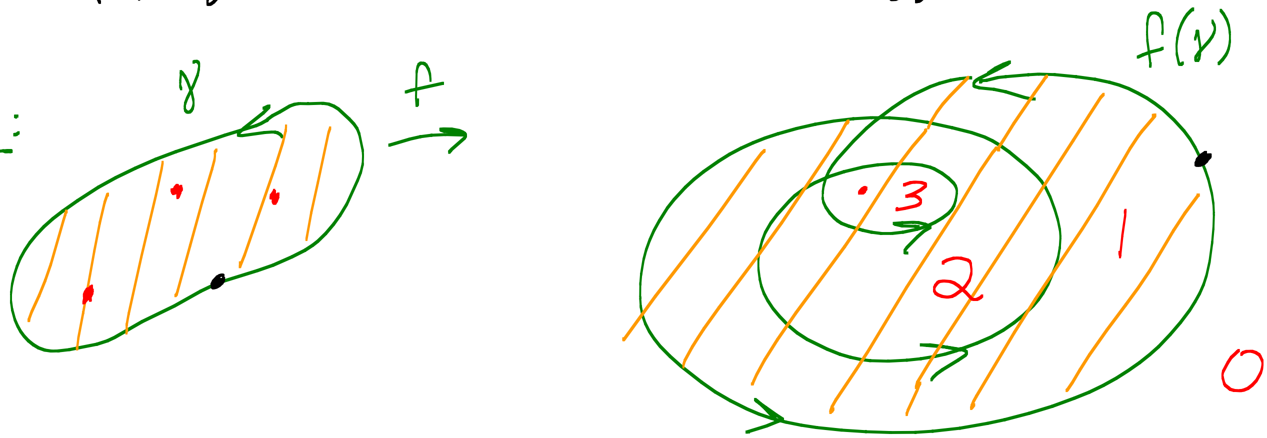
Possible: 2 separate simple zeroes or <u>one</u> double zero. (no poles assumed)

<u>Great fact</u>: Replace $\frac{f'(z)}{f(z)}$ by $\frac{f'(z)}{\underbrace{f(z)-w_0}_{F(z)}} = \frac{F'(z)}{F(z)}$

<u>Def^n</u>: "$f(z)=w_0$ with mult m at z_0" means

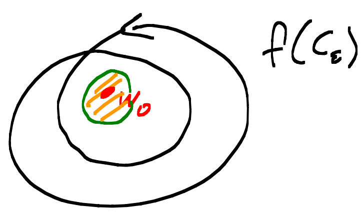
$f(z)-w_0$ has a zero of mult m at z_0.

<u>Picture</u>:



Consequence: <u>Open mapping theorem</u>: If f is a non-constant analytic function, the f maps open sets to open sets!

Shrink ε so that z_0 is only possible zero of $f(z)-f(z_0)$ inside C_ε

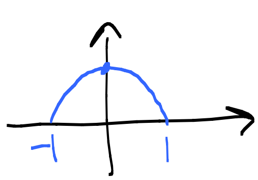


Everything inside small disc about w_0 gets hit (same # times).

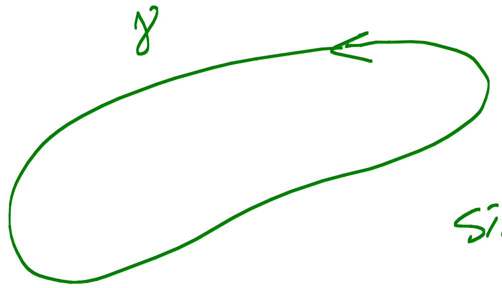
<u>Way fake $\mathbb{R}\to\mathbb{R}$</u>: $h(t) = 1-t^2$ $\quad h:(-1,1) \longrightarrow (0,1]$

$(-1,1)$: open

$(0,1]$: not open



Rouché's Theorem:



f and h analytic inside and on simple (counterclockwise) curve γ.

If $|h(z)| < |f(z)|$ along γ, then

f and $f+h$ have same # zeroes inside γ

(counted with mult.).

Feeling:

Hyp: leash length < dist of man to pole.

Dog has to go around pole same # times as man.

Arg Princ $\Rightarrow$ f and $f+h$ have same # zeroes!