Lesson 31 Harmonic functions again

Rouché's Theorem: Suppose f and h are analytic inside and on a simple closed curve γ and $|h(z)| < |f(z)|$ on γ. Then f and $f+h$ have same number of zeroes (counted with mult) inside γ.

EX: $\underbrace{z^3 + 9z}_{h(z)} + \underbrace{27}_{f(z)}$ on $D_2(0)$. How many zeroes

$\leftarrow$ the biggest

Note: $|h(z)| \le |z^3| + |9z| = \underbrace{2^3 + 9\cdot 2}_{26} $ on $C_2(0)$.

$26 < 27 = |f(z)|$ on $C_2(0)$.

So $|h(z)| < |f(z)|$ on $C_2(0)$.

Rouché's $\Rightarrow$ f and $f+h$ have same # zeroes inside, f has none. So $z^3 + 9z + 27$ also has none.

EX: Show that all the zeroes of $z^6 - 5z^2 + 10$ fall in the annulus $\{z : 1 < |z| < 2\}$.

On $C_2(0)$: $z^6 - 5z^2 + 10$

$\uparrow$

$|z^6| = 2^6 = 64$ on $C_2(0)$

Big one

$f(z) = z^6$

$h(z) = -5z^2 + 10$

Then $|h(z)| \le |5z^2| + 10$

$= 5\cdot 2^2 + 10 = 30$

$< 64 = |f(z)|$ on C_2

f has 6 zeroes inside $C_2(0)$. Hence, so does $f+h = z^6 - 5z^2 + 10$.

[f only has 6 zeroes on $\mathbb{C}$ by Fund. Thm. Alg.]

On $C_1(0)$: $\quad z^6 - 5z^2 + \underbrace{10}_{=f(z)}$ ← the big one

Then $h(z) = z^6 - 5z^2$ and $|h(z)| \le |z^6| + |5z^2|$

$$= 1^6 + 5 \cdot 1^2$$

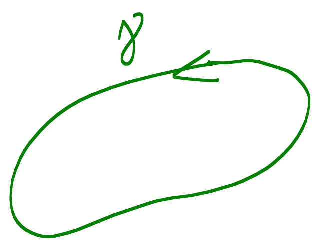
$$= 6 < 10 = f(z) \text{ on } C_1$$

So $|h| < |f|$ on $C_1(0)$. f has <u>no</u> zeroes. So

$z^6 - 5z^2 + 10 = f + h$ also has none.

So all 6 zeroes of poly fall inside C_2 and outside C_1. ✓

<u>Why Rouché's works:</u>



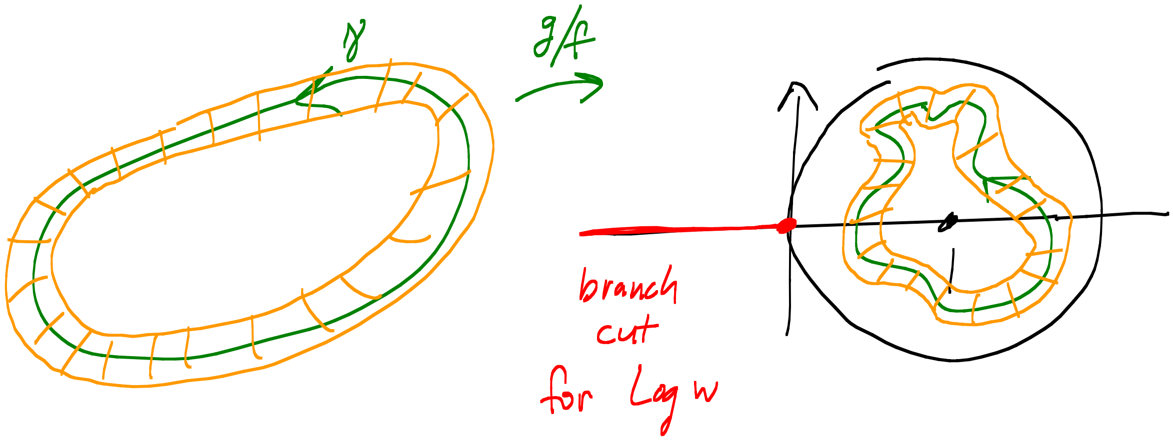
(*) $|h| < |f|$ on γ.

Let $\underbrace{g = f + h}$ ← want to show g & f have same # zeroes.

Divide by f

$$\frac{g}{f} = 1 + \frac{h}{f}$$

$$\left| 1 - \frac{g}{f} \right| = \left| \frac{h}{f} \right| < 1 \text{ on } \gamma \text{ by } (*)$$



Cool thing: $F = \frac{g}{f}$. Then $\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}$

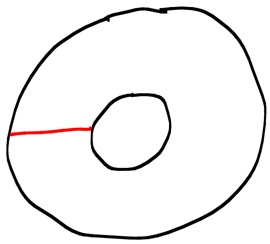
Aha! $\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{F'}{F}\,dz}_{\frac{1}{2\pi i}\int_\gamma \frac{d}{dz}(\operatorname{Log} F)\,dz \;=\; 0} = \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\,dz}_{\#\text{ zeroes } g} - \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz}_{\#\text{ zeroes } f}$

Start reading Chap 7.

Fact: u is harmonic on a domain Ω if and only if it is locally the real part of an analytic fcn, meaning that, for each $z_0 \in \Omega$, there is a disc $D_\varepsilon(z_0) \subset \Omega$ and an analytic fcn f on $D_\varepsilon(z_0)$ such that $u = \operatorname{Re} f$ on $D_\varepsilon(z_0)$.

Note: If Ω is simply connected, then we can get an analytic f on all of Ω such that $u = \operatorname{Re} f$ on Ω.

EX: $u = \operatorname{Ln}|z|$ on annulus $A = \{z : 1 < |z| < 2\}$ is harmonic.

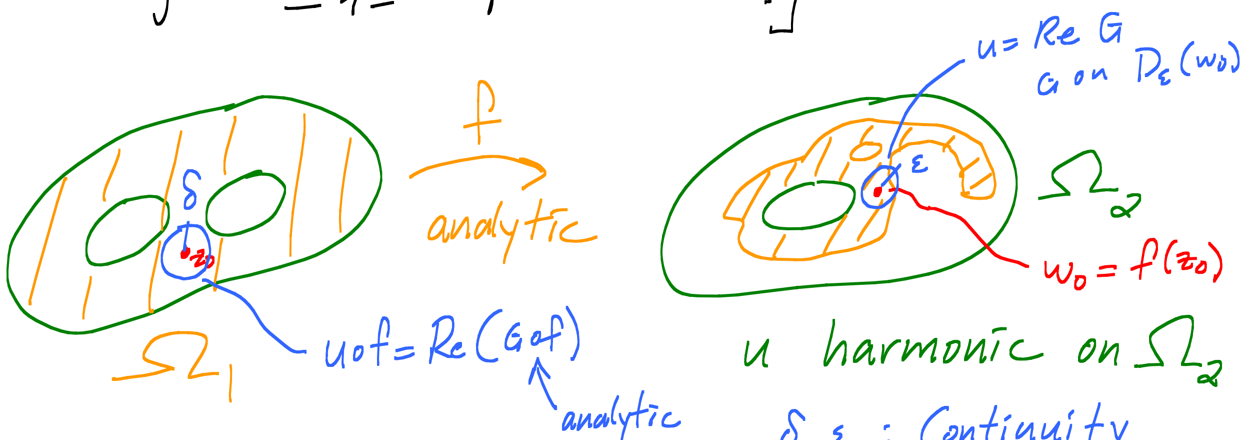


$\underbrace{\operatorname{Ln}|z| + i \operatorname{Arg} z}_{f(z)}$ works on $A - (-2,-1)$.

Can't make it work on A.

[Fact: Harm conj are unique up to $+C$.]

Thm: (7.1)



u harmonic on Ω_2

δ, ε : Continuity of analytic f at z_0

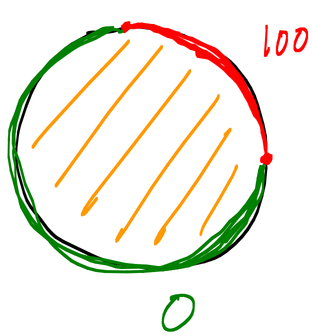
Then $u \circ f$ is harmonic on Ω_1.

"Harmonic composed with analytic are harmonic."

The hard way: $\Delta(u \circ f) = \ldots$ chain rule

... use C-R equs in right spots in the right way $= 0$

Exciting tool:

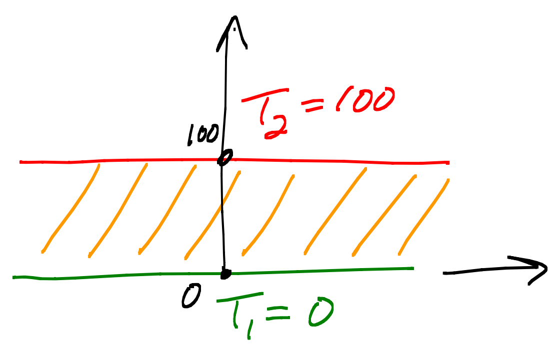


need f

analytic

1-1, onto

do right thing on boundary



$u = y = \operatorname{Im} z$