Review 1 for Exam 2 on Wed in class

Find Practice probs + old Exam 2 on home page.

Partial Fractions: Suppose $P(z)$ and $Q(z)$ are polys with no common factors and $\deg P < \deg Q$. Let the zeroes of Q be $a_1, a_2, \ldots, a_N$ and suppose the mult of a_k is m_k.

Then $$\frac{P(z)}{Q(z)} = \sum_{n=1}^{N} \sum_{k=1}^{m_k} \frac{A_{nk}}{(z-a_n)^k}$$

EX: $$\frac{z^4+1}{(z-i)^3(z+i)^2} = \left[\frac{A}{(z-i)^3} + \frac{B}{(z-i)^2} + \frac{C}{z-i}\right] + \left[\frac{D}{(z+i)^2} + \frac{E}{z+i}\right]$$

Remark: Freshman Calc: $$\frac{x^2+x+1}{(x^2+1)(x-7)} = \frac{A}{x-7} + \frac{Bx+C}{x^2+1}$$

$\underbrace{ax^2+bx+c}_{\text{real}}$ has complex roots: $= a(z-r)(z-\bar{r})$ ← conj of r

$$\frac{A}{z-r} + \frac{B}{z-\bar{r}} \leftarrow B = \bar{A} \quad \text{get Fresh Calc form.}$$

Proof of Partial Fractions:

Near a_n: $$\frac{P(z)}{Q(z)} = \frac{1}{(z-a_n)^{m_n}} \cdot \underbrace{\frac{P(z)}{a\prod_{k\neq n}(z-a_k)^{m_k}}}_{F_n(z) \text{ analytic near } a_n}$$

$$= A_0 + A_1(z-a_n) + \cdots$$

$$= \underbrace{\frac{A_0}{(z-a_n)^{m_n}} + \cdots + \frac{A_{m_n-1}}{z-a_n}}_{R_n(z) \leftarrow \text{princ. part.}} + (\text{power series})$$

Define $\displaystyle H(z) = \frac{P(z)}{Q(z)} - \sum_{n=1}^{N} R_n(z)$

Claim: H has removable sing at zeroes of Q

Near a_n: $\displaystyle H(z) = \underbrace{\left(\frac{P}{Q} - R_n(z)\right)}_{\text{removable sing at } a_n} - \underbrace{\sum_{k\neq n} R_k(z)}_{\text{analytic near } a_n}$

Give $H(z)$ the proper values at the a_n's to make it entire.

Claim: $\displaystyle \lim_{z\to\infty} H(z) = 0$. So H is a bnd entire fcn.

Why: $A|z|^{\deg P} \le |P(z)| \le B|z|^{\deg P}$ if $|z| > R_P$

$a|z|^{\deg Q} \le |Q(z)| \le b|z|^{\deg Q}$ if $|z| > R_Q$

So $\displaystyle \left|\frac{P(z)}{Q(z)}\right| \le \frac{B}{a} \frac{1}{|z|^{\underbrace{\deg Q - \deg P}_{>0}}}$ if $|z| > \mathrm{Max}(R_P, R_Q)$

and $\frac{P}{Q} \to 0$ as $z\to\infty$.

Aha! $\displaystyle \frac{A_{nk}}{(z-a_n)^k} \to 0$ as $z\to\infty$ too!

So $H(z)\to 0$ as $z\to\infty$. H bnd entire.

Liouville's $\Rightarrow$ $H \equiv$ const. Limit at ∞ $\Rightarrow$ const $= 0$.

So $\frac{P}{Q} = \sum R_n \leftarrow$ partial fractions!

Finding residues:

Attempt #1: $$\mathrm{Res}_a \frac{f(z)}{z-a} = f(a) \quad \text{if } f \text{ analytic near } a.$$

EX: $$\frac{e^z}{z^2+4} = \frac{e^z}{(z-2i)(z+2i)} = \frac{e^z/(z+2i)}{z-2i}$$

$$\mathrm{Res}_{2i} = \frac{e^{2i}}{2i+2i} = \frac{1}{4i}e^{2i}$$

EX: $$\frac{e^z}{z^4+1} = \frac{e^z}{(z-r_1)(z-r_2)(z-r_3)(z-r_4)} \quad \dots \text{ ugh!}$$

Attempt #2: $$\mathrm{Res}_a \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)} \quad \text{when } g \text{ has a } \underline{\text{simple}} \text{ zero at } a.$$

$$\mathrm{Res}_{e^{i\pi/4}} \frac{e^z}{z^4+1} = \frac{e^z}{4z^3}\bigg|_{e^{i\pi/4}} = \frac{e^{\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{4e^{i3\pi/4}}$$

Attempt #3: Use power series and algebra.

EX: $$\frac{z^\alpha}{(z^2+4)^2} = \frac{z^\alpha}{(z-2i)^2(z+2i)^2}$$

$$= \frac{1}{(z-2i)^2}\left[\underbrace{\frac{z^\alpha}{(z+2i)^2}}\right]$$

$$H(z) = A_0 + A_1(z-2i) + A_2(z-2i)^2 + \dots$$

$$= \underline{\frac{A_0}{(z-2i)^2} + \frac{A_1}{z-2i}} + \text{power series}$$

princ. part.

$$\text{Res}_{2i} = A_1 = \frac{H'(2i)}{1!} \quad \text{by Taylor's}$$

$z^\alpha = e^{\alpha \log z}$ ← take princ. branch for this prob.

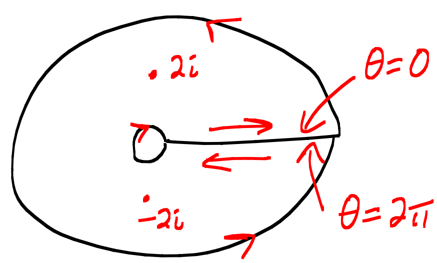
$$\frac{d}{dz} z^\alpha = \underbrace{\left(e^{\alpha \log z}\right)\left(\alpha \cdot \frac{1}{z}\right)}_{} = \frac{\alpha z^\alpha}{z} \overset{?}{=} \alpha z^{\alpha-1}$$

$$\alpha\, e^{\alpha \operatorname{Log} z} \cdot \frac{1}{e^{\operatorname{Log} z}}$$

$$= \alpha\, e^{\alpha \operatorname{Log} z - \operatorname{Log} z}$$

$$= \alpha\, e^{(\alpha-1)\operatorname{Log} z} = \alpha z^{\alpha-1} \quad \checkmark$$

$$\int_0^\infty \frac{x^\alpha}{(x^2+4)^2}\,dx$$



$z^\alpha = e^{\alpha \log z}$

Great practice problem: What is the residue at a of $\frac{f(z)}{g(z)}$ if f and g are analytic near a and g has a double zero at a?

$$\left.\begin{array}{l} g(a) = 0 \\ g'(a) = 0 \\ g''(a) \neq 0 \end{array}\right\} \text{double}$$

Hmmm. $$g(z) = \overset{=0}{a_0} + \overset{=0}{a_1}(z-a) + a_2(z-a)^2 + \cdots$$

$$= (z-a)^2 \Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{G(z)} \Big]$$

Note: $a_2 = G(a)$ $\quad a_3 = \frac{G'(a)}{1!}$

$a_2 = \frac{g''(a)}{2!}$ $\quad a_3 = \frac{g'''(a)}{3!}$

$$\frac{f(z)}{g(z)} = \frac{1}{(z-a)^2}\left[\underbrace{\frac{f(z)}{G(z)}}\right]$$

$H(z)$ nice and analytic near a!

$= A_0 + A_1(z-a) + \cdots$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \text{power series}$$

$$\operatorname{Res}_a \frac{f}{g} = A_1 = \frac{1}{1!}H'(a) = \left.\frac{f'G - fG'}{G^2}\right|_{z=a}$$

$G(a) = a_2$

$G'(a) = a_3$

$$= \frac{f'(a)\left[\frac{g''(a)}{2!}\right] - f(a)\left[\frac{g'''(a)}{3!}\right]}{\left[\frac{g''(a)}{2!}\right]^2}$$

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