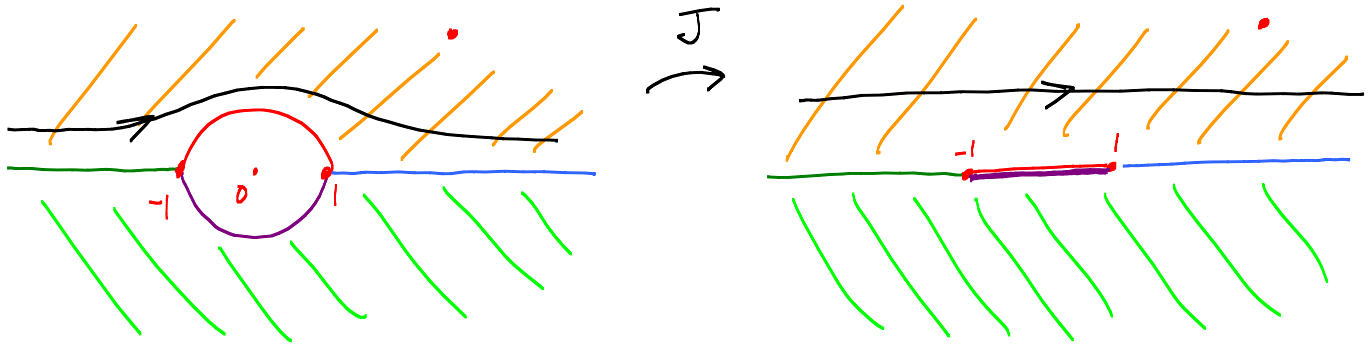
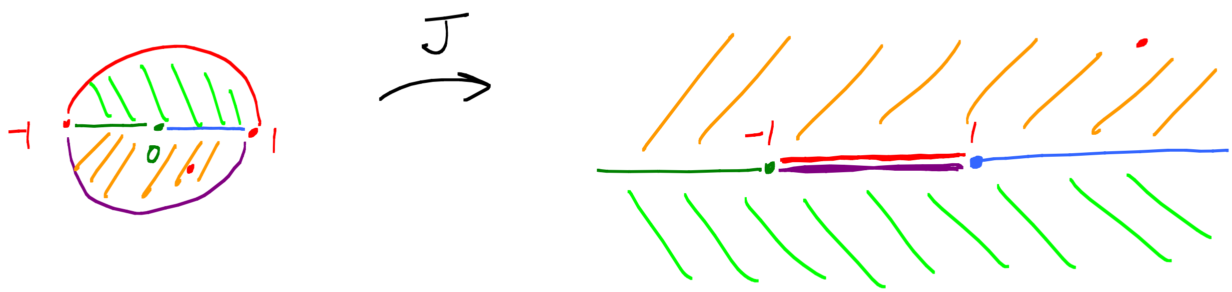
# Lesson 35 Linear Fractional Transformations 7.3

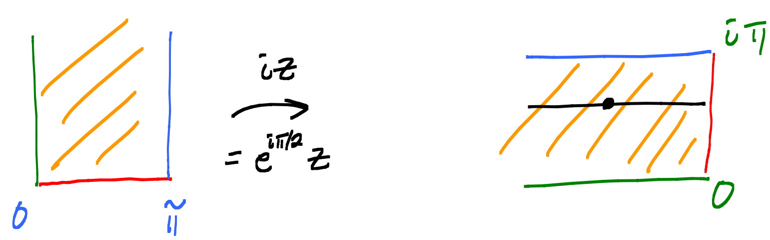
Jukovsky Map: $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$




## Complex cosine as a conformal map

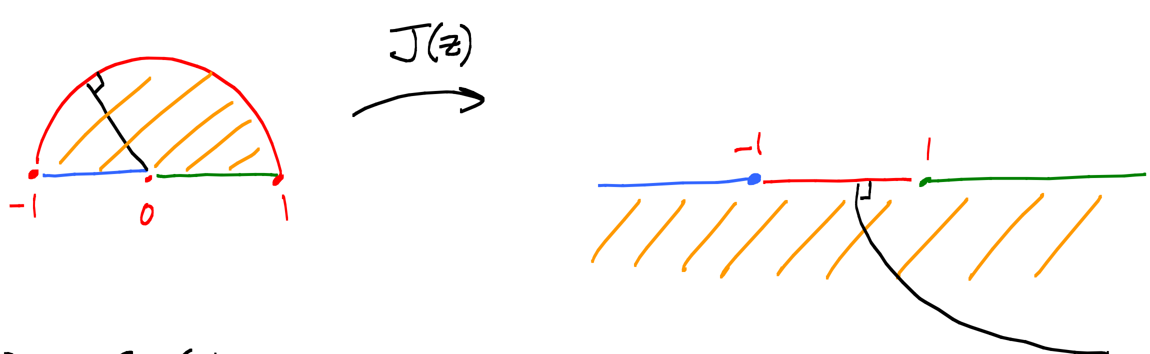
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right) = J(e^{iz})$$




$e^z = e^x e^{iy}$

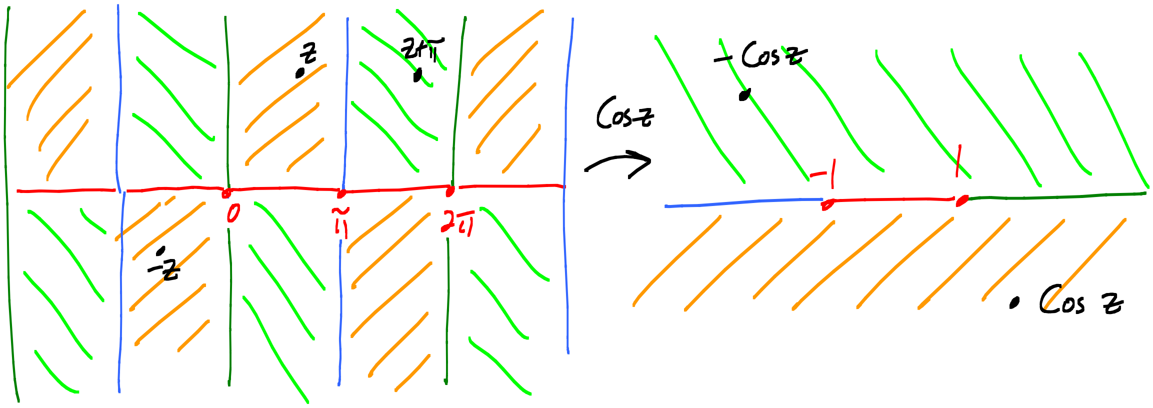
$-\infty < x < 0$

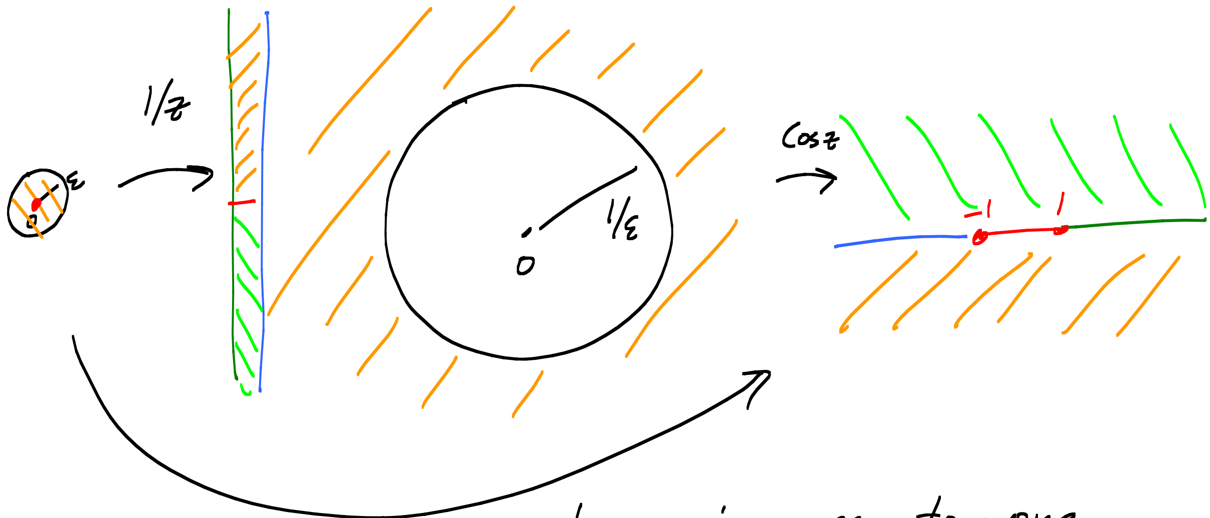
$0 < e^x < 1$




$\cos(-z) = \cos(z)$

$\cos(z + 2\pi) = \cos(z)$ $\qquad$ $\cos(z + \pi) = -\cos(z)$




$\cos \frac{1}{z}$ is $\infty$-to-one and hits all of $\mathbb{C}$

Conclude: $0$ is an essential sing of $\cos \frac{1}{z}$

i.e., $\infty$ is an ess sing of $\cos z$

What about $\sin z$: $\sin z = \cos\left(z - \frac{\pi}{2}\right)$ Graph just shifted for $\sin z$.

Linear Fractional Transformations: LFT's $\leftarrow$ me

Book: Möbius Transf

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Me: $\frac{z-a}{1-\bar{a}z}$ is a Möb. Trans.

Why: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$. Dumb cases: $c=d=0$ ouch!
$a=b=0$ dumb!

$\det = 0$ implies one row = multiple of other

$\Longrightarrow$ $L(z) \equiv$ const. Dumb.

Step 1: Long division:

$$\begin{array}{r} \frac{a}{c} \\ cz+d \overline{\big) az+b} \\ \underline{az+\frac{ad}{c}} \\ \left(b-\frac{ad}{c}\right) \end{array}$$

$c=0$
$L(z)=Az+B$
Assume
$c\neq 0$

$$L(z) = \underbrace{\frac{a}{c}}_{B} + \frac{\overbrace{\left(b-\frac{ad}{c}\right)}^{Re^{i\alpha}}}{\underbrace{c}_{c=re^{i\theta}}z+d}$$

Step 2: 4 Basic maps:

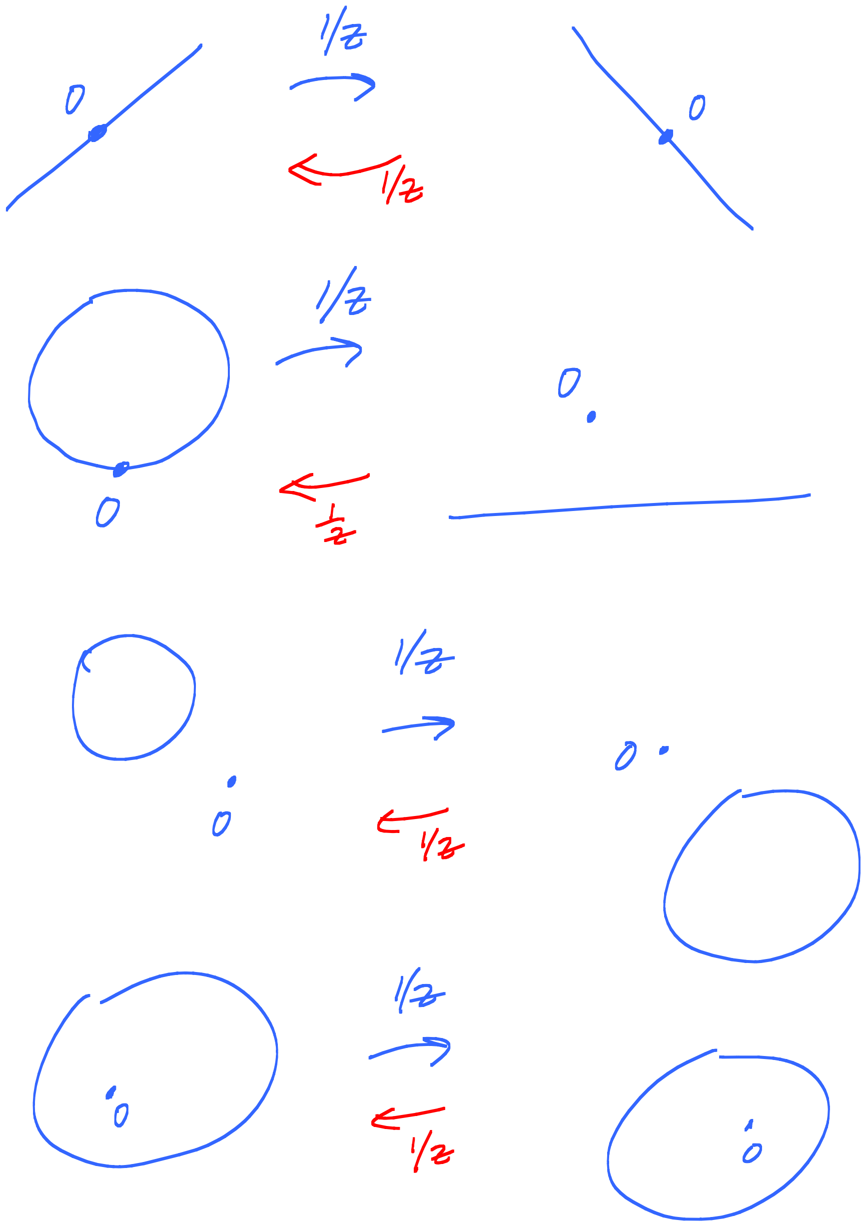
1) $z \mapsto rz$ $(r>0)$ Stretch
2) $z \mapsto e^{i\theta}z$ Rotation
3) $z \mapsto z+B$ $(B\in\mathbb{C})$ Translation
4) $z \mapsto \frac{1}{z}$ Inversion

Seven steps: Stretch by $r$, rotate by $\theta$, translate by $d$, invert, stretch by $R$, rotate by $\alpha$, translate by $B$.

Fact: 1) LFT's given by compositions of Basic maps.

2) Basic maps 1,2,3 $\{\text{lines}\} \to \{\text{lines}\}$
$\{\text{circles}\} \to \{\text{circle}\}$

Inversion $\{\text{lines, circles}\} \to \{\text{lines, circle}\}$ !




Fact: LFT's : {lines, circles} $\longrightarrow$ {lines, circles}

Aha! Three pts determine a circle. Two pts determine line
(+ they shoot out to $\infty$)

Fact: On Riemann sphere: {lines, circles} are slices of sphere