# Lessons 35-36 Linear Fractional Transformations (LFT's)

$L(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

**Exam 2**

Ave 70/100

| | |
|---|---|
| 80-100 | A |
| 60-79 | B |
| 40-59 | C |
| 30-39 | D |
| <30 | F |

Basic maps:

1. $z \mapsto rz$ stretch
2. $z \mapsto e^{i\theta} z$ rot.
3. $z \mapsto z+b$ trans.
4. $z \mapsto 1/z$ inv.

1, 2, 3 : $\mathbb{C} \underset{\text{onto}}{\xrightarrow{1-1}} \mathbb{C}$, $\infty \mapsto \infty$ $\leftarrow$ $\lim_{z \to \infty} L(z) = \infty$

$\lim_{|z| \to \infty} |L(z)| = \infty$

Let $L(\infty) = \infty$

4. $\mathbb{C} - \{0\} \underset{\text{onto}}{\xrightarrow{1-1}} \mathbb{C} - \{0\}$ $0 \mapsto \infty$, $\infty \mapsto 0$ Say $L(0) = \infty$, $L(\infty) = 0$

Fact: LFT's are 1-1 onto self maps of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Case: ($c = 0$) $L(z) = Az + B$ $\infty \mapsto \infty$

Case: $c \neq 0$ $L(z) = \frac{az+b}{cz+d} = \frac{a + \frac{b}{z}}{c + \frac{d}{z}}$

$\lim_{z \to \infty} L(z) = \frac{a}{c}$ and a simple pole at $z = \frac{-d}{c}$

Let $L\left(\frac{-d}{c}\right) = \infty$ and $L(\infty) = \frac{a}{c}$

## Inverse of $L(z)$:

$$w = \frac{az+b}{cz+d} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det \neq 0$$

$$cwz + dw = az + b$$

$$z(cw - a) = -dw + b$$

$$z = \frac{-dw+b}{cw-a} = \frac{dw-b}{-cw+a} \qquad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Fact: $L^{-1}(w) = \dfrac{dw-b}{-cw+a}$

Cool Fact: $$L_1(L_2(z)) = \frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_1} = \frac{Az+B}{Cz+D}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbb{A}$$

$$L^{-1} \sim \mathbb{A}^{-1}$$

$$L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2$$

"Group homorphism"

Fact: LFT's form a "group".

$L_1 \oplus L_2 \sim L_1 \circ L_2$

id : $L(z) = z = \dfrac{1 \cdot z + 0}{0 \cdot z + 1}$

$L_1^{-1}$ is the inverse.

Note: $f$ has a pole of order $m$ at $a$:

$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + \text{p.s.}$$

$$(z-a)^m f(z) = A_m + \cdots + A_1 (z-a)^{m-1} + \cdots$$

$a$ is removable. Call analytic fcn $F = (z-a)^m f$

$$A_1 = \frac{F^{(m-1)}(a)}{(m-1)!} = \lim_{z \to a} \frac{F^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{(m-1)}} \left[ (z-a)^m f(z) \right]$$