Lesson 36 LFTs $\qquad L(z) = \dfrac{az+b}{cz+d}, \quad ad-bc \neq 0$

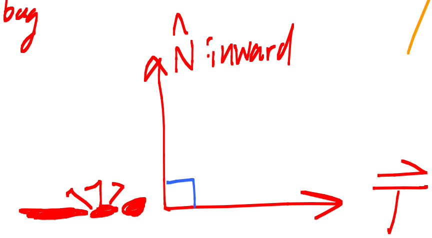
HWK 11 due Fri

Facts: 1) LFTs : {lines, circles} $\longrightarrow$ {lines, circles}

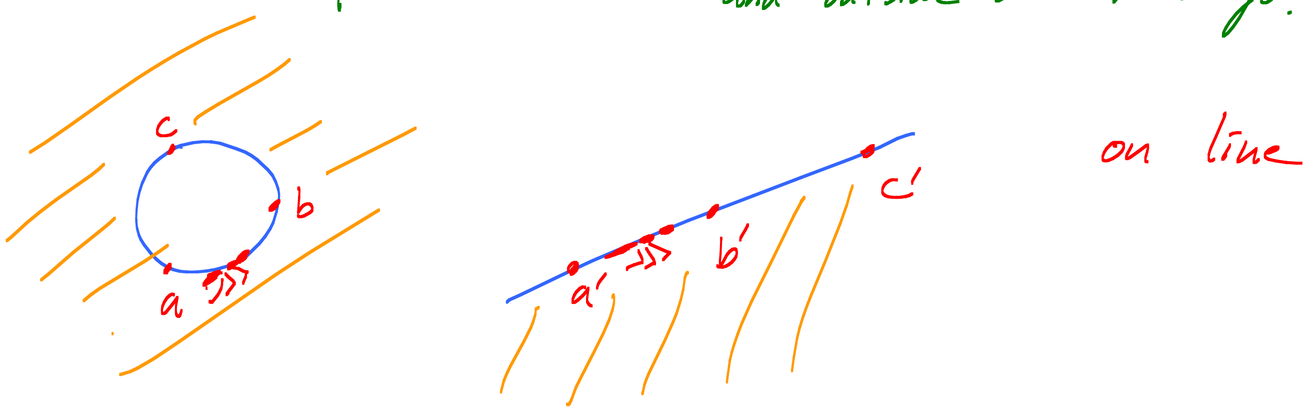
2) : $\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$

3) Analytic and $L'(z) \neq 0$, so they are conformal.

EX:



Two tests: Bug test. One point test. } to determine where inside and outside of circles go.



Big Fact: A LFT that fixes 3 pts must be the identity.

Why: $L(z_j) = z_j \qquad j = 1,2,3 \qquad\qquad z_j = \dfrac{az_j+b}{cz_j+d}$

$$c z_j^2 + (a-d) z_j - b = 0$$

Quadratic eqn with 3 zeroes! Must be the <u>zero</u> quadratic!

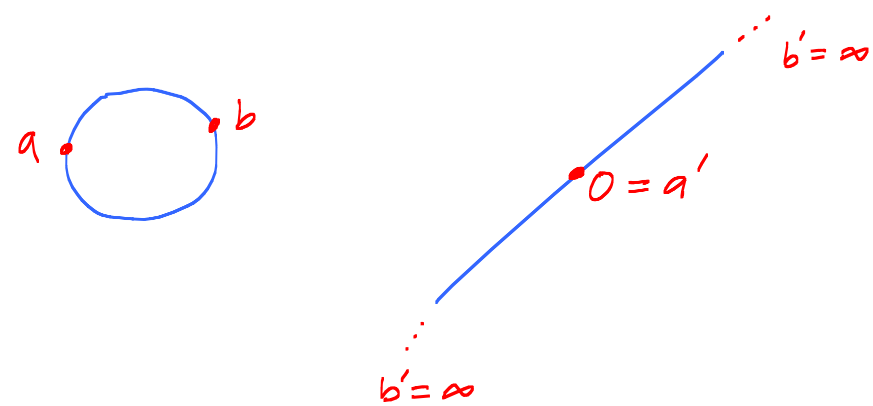
Must have $c=0$, $a-d=0$, $b=0$ $ad - \underbrace{bc}_{=0} \neq 0$

So $a \neq 0, d \neq 0$.

Now $L(z) = \dfrac{az + 0}{0 \cdot z + d} = \left(\dfrac{a}{d}\right) z = 1 \cdot z$ ✓

<u>Cor</u>: If two LFTs agree at 3 pts, they must be the same.

<u>How to cook up LFTs</u>: $L(z) = \dfrac{z-a}{z-b}$ $a \mapsto 0$, $b \mapsto \infty$



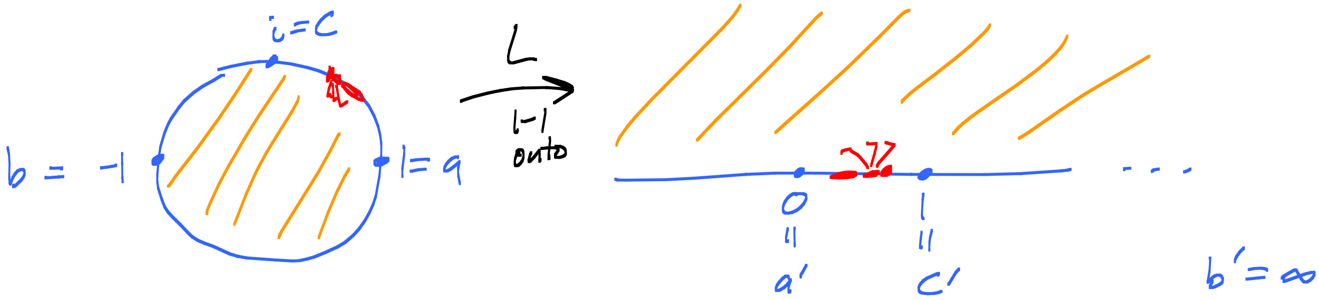
Hmmm. What if I wanted $L(c) = 1$ too?

$$\tilde{L}(c) = K \cdot \frac{(c)-a}{(c)-b} \quad \text{where } K = \frac{c-b}{c-a}$$

<u>Fact</u>: $L(z) = \dfrac{c-b}{c-a} \cdot \dfrac{z-a}{z-b}$

$a \mapsto 0$
$b \mapsto \infty$
$c \mapsto 1$

Prob:



$$L(z) = \frac{i+1}{i-1} \cdot \frac{z-1}{z+1}$$

Prob: Find an LFT:

$$z_1 \mapsto w_1$$
$$z_2 \mapsto w_2$$
$$z_3 \mapsto w_3$$

(3 pts determine a circle (or line))

L_1: $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$

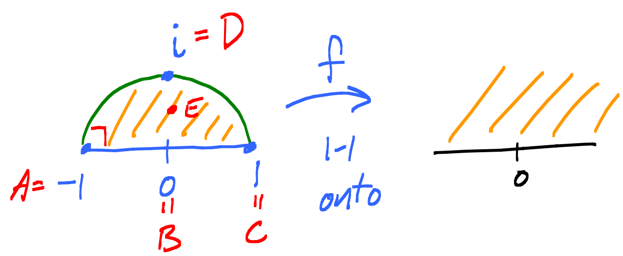
$$L_1(z) = \frac{z_2 - z_3}{z_2 - z_1} \cdot \frac{z - z_1}{z - z_3}$$

L_2: $w_1 \mapsto 0$, $w_2 \mapsto 1$, $w_3 \mapsto \infty$

$L_2(w) =$ with w's in place of z's.

Aha! $L_2^{-1}(L_1(z))$ does it! It's unique.

Prob: Find a conformal map of



Hmmm. Use an LFT to blow $z=1$ out to ∞.

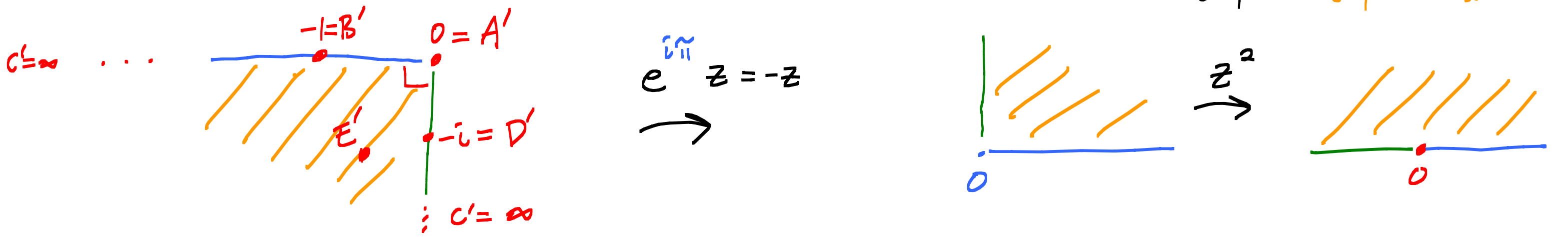
$$L(z) = \frac{z+1}{z-1}$$

$L(1) = \infty$

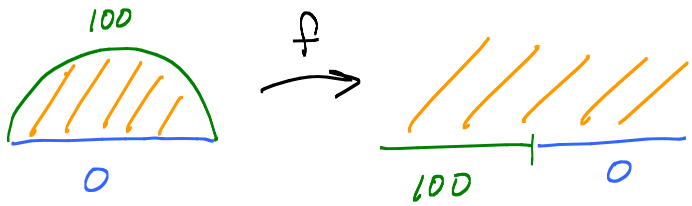
$L(-1) = 0$

$L(0) = \frac{0+1}{0-1} = -1$

$L(i) = \frac{i+1}{i-1} \cdot \frac{-i-1}{-i-1} = \frac{-2i}{2} = -i$



Ans: $f(z) = \left(-\left[\dfrac{z+1}{z-1}\right]\right)^2 = \left(\dfrac{z+1}{z-1}\right)^2$

Prob: 

$\dfrac{100}{\pi}\operatorname{Arg} z$

Aha! $\dfrac{100}{\pi}\operatorname{Arg}\left(\dfrac{z+1}{z-1}\right)^2$ solves prob on the half disc!

$$T(x,y) = \frac{100}{\pi}\operatorname{Tan}^{-1}\left(\frac{\operatorname{Im}\left(\frac{x+iy+1}{x+iy-1}\right)^2}{\operatorname{Re}\left(\frac{x+iy+1}{x+iy-1}\right)^2}\right)$$

get fcn of x and y.