Lesson 6 Cauchy-Riemann Equations

Lessons 4,5,6 due Wed

Theorem: If u and v are $\mathcal{C}^1$-smooth on a domain Ω and satisfy the CR Eqns:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{on } \Omega, \text{ then}$$

$f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic on Ω.

Taylor's thm: u, v $\mathcal{C}^1$-smooth

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + R_u(x,y)$$

$$v = v^0 + v_x^0(x-x_0) + v_y^0(y-y_0) + R_v$$

where
$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (z \to z_0)}} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \longleftarrow \sqrt{} = |z-z_0|$$

Pf of CR Eqns Thm: $DQ = \dfrac{f(z) - f(z_0)}{z - z_0}$ $\quad f(z_0) = u^0 + iv^0$

$z = x+iy$

$z_0 = x_0 + iy_0$

$$DQ = \frac{\left[u_x^0(x-x_0) + \overset{=-v_x^0}{u_y^0}(y-y_0)\right] + i\left[v_x^0(x-x_0) + \overset{=u_x^0}{v_y^0}(y-y_0)\right]}{z-z_0} + \underbrace{\frac{R_u + iR_v}{z-z_0}}_{\mathcal{R}}$$

Aha! This makes the numerator a complex product:

$$= \frac{\left[u_x^0 + i v_x^0\right]\cdot\left[\overbrace{(x-x_0) + i(y-y_0)}^{z-z_0}\right]}{z-z_0} + \mathcal{R}$$

Red box #1!

$$= \boxed{u_x^0 + i v_x^0} + \mathcal{R}$$

where $|\mathcal{R}| = \dfrac{|R_u + iR_v|}{|z - z_0|} \le \dfrac{|R_u|}{|z-z_0|} + \dfrac{|R_v|}{|z-z_0|}$

$\xrightarrow[\text{by Taylor's}]{} 0$ as $z \to z_0$

So $f'(z_0)$ exists and = red box #1.

Cor: e^z is complex diff'ble on $\mathbb{C}$ and $\frac{d}{dz}(e^z) = e^z$.

$\mathbb{R}^2 \to \mathbb{R}^2$	$\mathbb{C} \to \mathbb{C}$
$(x,y) \mapsto (u(x,y), v(x,y))$	$z \mapsto f(z)$

Jacobian matrix

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$\underbrace{= \begin{bmatrix} A & -B \\ B & A \end{bmatrix}}_{\text{linear map represents complex multiplication by } [A+iB]} \leftarrow \text{C-R Eqns}$$

linear map represents complex multiplication by $[A+iB]$

$f'(z)$ exists

$$\boxed{f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)}$$

where $\dfrac{R(z)}{|z-z_0|} \to 0$ as $z \to z_0$.

Exciting consequence: Suppose u, v are C^2-smooth and satisfy the C-R Eqns (so $u+iv$ is analytic). Then u and v must be harmonic fcns

$$\left[\text{meaning that } \Delta u \equiv 0,\ \Delta v \equiv 0 \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]$$

Why:

$$u_x = v_y \quad (A)$$

$$u_y = -v_x \quad (B)$$

$$\frac{\partial}{\partial x}(A): \quad u_{xx} = v_{xy}$$

$$+ \quad \frac{\partial}{\partial y}(B): \quad u_{yy} = -v_{yx}$$

Aha! v C^2-smooth $\Rightarrow v_{xy} = v_{yx}$

$$\Delta u = 0 \checkmark$$

Similarly for v, use $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ ... etc

EX: $z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) \checkmark$ u, v harmonic

$$e^z = e^{x+iy} = (e^x \cos y) + i(e^x \sin y) \checkmark$$

Hwk Prob: $|e^z| = |e^{x+iy}| = |e^x e^{iy}| = \underbrace{|e^x|}_{e^x} \underbrace{|e^{iy}|}_{\sqrt{\cos^2 y + \sin^2 y} = 1}$

$$= e^x \leq 1 \quad \text{when } x \leq 0$$

Problem: Hmmm. If u is a given harmonic fcn, can I find a v such that $u+iv$ is analytic?

Yes! v is called a harmonic conjugate for u.

Sol^n: Given u, want v such that

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{array}{l} v_x = -u_y \\ v_y = u_x \end{array}$$

Want v with $\nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} \leftarrow \vec{F}$

Aha! v is a potential fcn for this $\vec{F}$!

[We need $\operatorname{Curl} \vec{F} = 0$ (in $\mathbb{R}^3$)]

In $\mathbb{R}^2$, the condition is:

$$\nabla u = \vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \qquad \begin{array}{ll} u_x = F_1 & (A) \\ u_y = F_2 & (B) \end{array}$$

It must be that $0 = \underbrace{\frac{\partial}{\partial y}(A)}_{u_{yx}} - \underbrace{\frac{\partial}{\partial x}(B)}_{u_{xy}} = (F_1)_y - (F_2)_x$

$u_{yx} = u_{xy} \leftarrow =$ when u C^2

Math 261: Necessary and Sufficient condition for u to exist is $\boxed{(F_1)_y = (F_2)_x}$

when Ω has no holes!

Do we have this $(\underbrace{F_1}_{-u_y})_y \stackrel{?}{=} (\underbrace{F_2}_{u_x})_x$

$-u_{yy} \stackrel{?}{=} u_{xx}$ ✓ $\Delta u \equiv 0.$

EX: $u(x,y) = x^2 - y^2 + xy + 3$ $\quad \Delta u \equiv 0$ ✓

Laplacian

Find v with
$$\begin{cases} v_x = -u_y = -\frac{\partial}{\partial y}(x^2 - y^2 + xy + 3) = 2y - x & (A) \\ v_y = u_x = 2x + y & (B) \end{cases}$$

Step 1: Use (A):

$$v = \int 2y - x \, dx = 2xy - \frac{1}{2}x^2 + \underbrace{C(y)}$$

treat y like a const.

arbitrary fcn of y

(A is used up.)

Step 2: Use (B): $\frac{\partial v}{\partial y} = 2x + y$

$$\frac{\partial}{\partial y}\left(2xy - \frac{1}{2}x^2 + C(y)\right) = 2x + y$$

$$2x - 0 + C'(y) = 2x + y$$

Miracle! all x's cancel!

$$C'(y) = y$$

So $C(y) = \frac{1}{2}y^2 + K$

Get $\boxed{v = 2xy - \frac{1}{2}x^2 + \left(\frac{1}{2}y^2 + K\right)}$

What is $u + iv$ as a fcn of z?

Trick: $z = x + iy$, $\bar{z} = x - iy$

$$\begin{cases} x = \dfrac{z + \bar{z}}{2} \\ y = \dfrac{z - \bar{z}}{2i} \end{cases}$$

Write out $u + iv$ = sum of x, y stuff = sum of $z, \bar{z}$ stuff

... all $\bar{z}$'s cancel and go away.

$f(z)$ = poly in z!