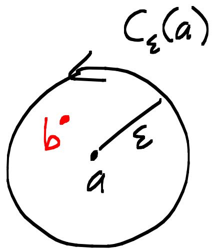
Lesson 14 The Cauchy Integral Formula

12,13,14 due Wed

Exam 1 in class Mon., Sept. 30

Nice fact:



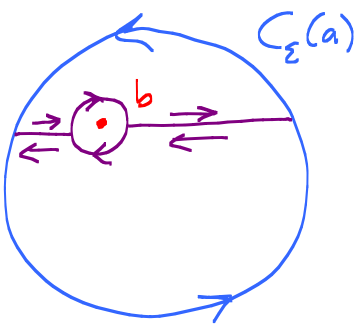
$C_\varepsilon(a)$: $z(t)=a+\varepsilon e^{it}$, $0\le t\le 2\pi$

$$\int_{C_\varepsilon(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{(a+\varepsilon e^{it})-a}\, i\varepsilon e^{it}\,dt$$

$$= \int_0^{2\pi} i\,dt = 2\pi i$$

Also:
$$\int_{C_\varepsilon(a)} \frac{1}{z-b}\,dz = \begin{cases} 2\pi i & b \text{ inside } C_\varepsilon(a) \\ 0 & b \text{ outside} \end{cases}$$

by Cauchy Thm.



Cauchy Thm $\Longrightarrow$

$$0 = \underbrace{\int_{\gamma_{top}} \frac{1}{z-b}\,dz}_{0} + \underbrace{\int_{\gamma_{bot}} \frac{1}{z-b}\,dz}_{0}$$

$$0 = \int_{C_\varepsilon(a)} \frac{1}{z-b}\,dz - \int_{C_{purple}} \frac{1}{z-b}\,dz$$

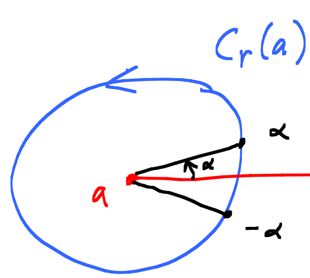
$= 2\pi i$

clockwise

= − counterclw

So
$$\int_{C_\varepsilon(a)} \frac{1}{z-b}\,dz = 2\pi i$$

Another way:



C_α starts at $a+re^{i\alpha}$
ends at $a+re^{i(2\pi-\alpha)}$

branch cut
for $\log_0(z-a)$

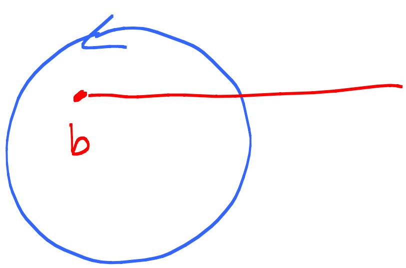
$$\int_{C_r(a)} \frac{1}{z-a}\,dz = \lim_{\alpha \to 0} \int_{C_\alpha} \frac{1}{z-a}\,dz$$

$$= \lim_{\alpha \to 0} \int_{C_\alpha} \frac{d}{dz}\left[\log_0(z-a)\right] dz$$

$$= \lim_{\alpha \to 0} \left[\log_0(z-a) \Big|_{a+re^{i\alpha}}^{a+re^{i(2\pi-\alpha)}} \right]$$

$$= \lim_{\alpha \to 0} \left[\left(\operatorname{Ln} r + i(2\pi - \alpha) \right) - \left(\operatorname{Ln} r + i\alpha \right) \right]$$

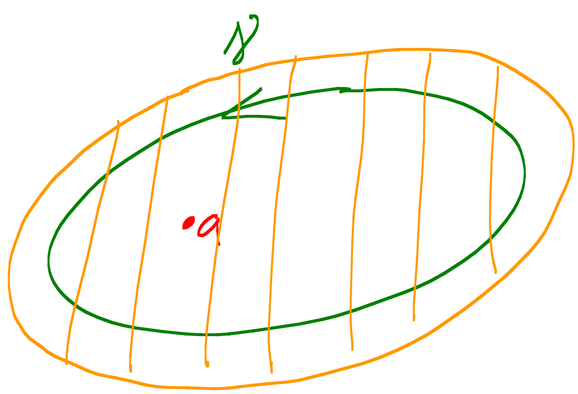
$$= \lim_{\alpha \to 0} i(2\pi - 2\alpha) = 2\pi i \quad \checkmark$$



$\log_0(z-b)$

Same calculation for b not center works.

Cauchy Integral Formula

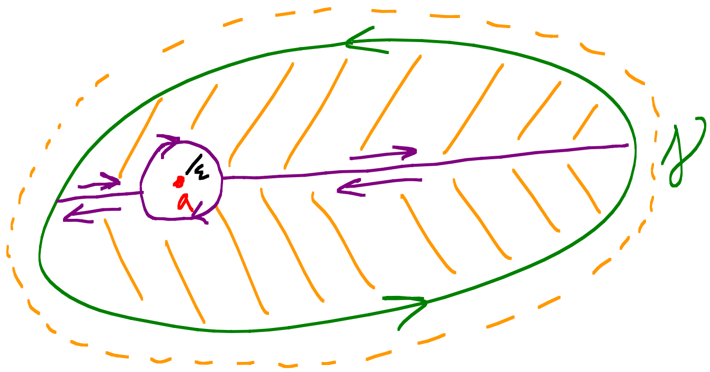


Assume f is analytic inside and on a simple closed curve γ in counterclockwise sense and a is inside γ.

Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$

Why:



Cauchy's Thm $\Longrightarrow$

$$0 = \underbrace{\int_{\gamma_{top}} \frac{f(z)}{z-a}\,dz}_{=0} + \underbrace{\int_{\gamma_{bot}} \frac{f(z)}{z-a}\,dz}_{=0}$$

$$0 = \int_{\gamma} \frac{f(z)}{z-a}\,dz \;-\; \underbrace{\int_{C_\varepsilon(a)} \frac{f(z)}{z-a}\,dz}_{\longrightarrow\, 2\pi i\, f(a) \text{ as } \varepsilon \to 0.}$$

clockwise $= -$ccw

$$= \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it})-a}\, i\varepsilon e^{it}\,dt$$

$$= i\int_0^{2\pi} \underbrace{f(a+\varepsilon e^{it})}_{\longrightarrow f(a) \text{ as } \varepsilon \to 0}\,dt$$

because $f'(a)$ exists
$\Longrightarrow f$ cont. at a.

$$\longrightarrow i\int_0^{2\pi} f(a)\,dt = 2\pi i\, f(a)\ \checkmark$$

More carefully: $f(z) = f(a) + E(z)$ where $E(z) \to 0$ as $z \to a$.

$$\left| \int_0^{2\pi} f(a+\varepsilon e^{it})\,dt - \underbrace{\int_0^{2\pi} f(a)\,dt}_{2\pi f(a)} \right|$$

$$\leq \int_0^{2\pi} \Big| \underbrace{f(a+\varepsilon e^{it}) - f(a)}_{E(a+\varepsilon e^{it})} \Big|\,dt$$

Fantastic fact:

$$f(z) = \frac{1}{2\pi i} \underbrace{\int_{C_r(A)} \frac{f(w)}{w-z} \, dw}_{\text{infinitely complex diff'ble!}} \qquad z \text{ inside } C_r(a)$$

$$f \text{ analytic} \Rightarrow f' \text{ analytic} \Rightarrow f'' \text{ analytic} \Rightarrow \cdots \quad !$$

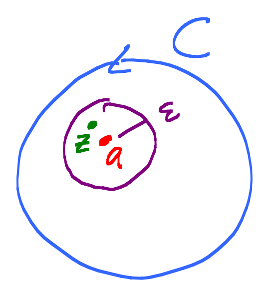
Why:

$$DQ = \frac{f(z) - f(a)}{z-a}$$

$$= \frac{1}{z-a} \cdot \frac{1}{2\pi i} \int_C \underbrace{\frac{f(w)}{w-z} - \frac{f(w)}{w-a}}_{f(w)\left[\frac{1}{w-z} - \frac{1}{w-a}\right]} \, dw$$

$$f(w) \frac{z-a}{(w-z)(w-a)} \xleftarrow{} \text{Aha! Pull out front!}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)(w-a)} \, dw$$



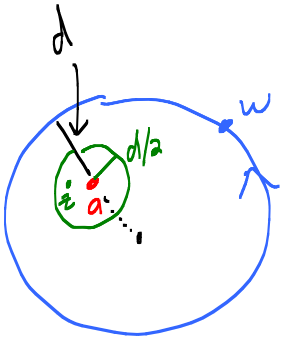
$$\xrightarrow{\text{expect}} \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^2} \, dw = I$$

Finally

$$|DQ - I| = \frac{1}{2\pi} \left| \int_C \underbrace{\frac{f(w)}{(w-z)(w-a)} - \frac{f(w)}{(w-a)^2}}_{f(w)\frac{z-a}{(w-z)(w-a)^2}} \, dw \right|$$

$\leftarrow$ pull out!

$$= \frac{1}{2\pi} \underbrace{|z-a|}_{\text{goes to zero}} \cdot \underbrace{\left| \int_C \frac{f(w)}{(w-z)(w-a)^2}\, dw \right|}_{\text{Just need to bound this}}$$



$$\leq \left(\max_{C} \frac{|f(w)|}{|(w-z)(w-a)^2|} \right) 2\pi r$$

$$\leq \left(\max_{w \in C} \frac{|f(w)|}{\left(\frac{d}{2}\right) \cdot d^2} \right) 2\pi r$$

f diff'ble $\Rightarrow$ f cont on C. $|f(A + re^{it})|$ is a cont. fcn on closed $[0, 2\pi]$. So it has a Max M.

Finally $|DQ - I| \leq |z-a| \cdot \dfrac{M}{\frac{d^3}{2}} \cdot r \longrightarrow 0$ as $z \to a$.

Thm: $$\frac{d}{dz} \int_\gamma \frac{\varphi(w)}{w-z}\, dw = \int \frac{\varphi(w)}{(w-z)^2}\, dw$$ ← "diff in z under $\int$"

when φ is continuous on γ.

Can repeat!

Higher order Cauchy Theorem:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}}\, dw$$