Lesson 16 Morera's Theorem, Maximum Principle

Big fact: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' ...

Why: Cauchy integral formula

Cor (Morera theorem) Given a continuous fcn $f: \Omega \to \mathbb{C}$ on a domain Ω. If $\int_\gamma f\,dz = 0$ for every closed curve γ in Ω, then f is analytic.

Why: We showed that if $\int_\gamma f\,dz = 0$ for any closed γ, then

$$F(z) = \int_{\gamma_a^z} f(w)\,dw \quad \text{is well defined}$$

and $F'(z) = f(z)$. Aha! F analytic $\Rightarrow$ $F' = f$ is analytic!

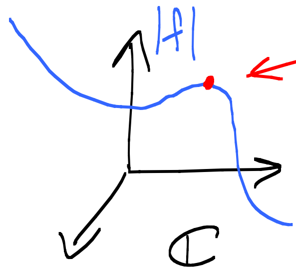
Cool thing: Morera's thm $\Rightarrow$ Complex power series converge to analytic fcns.

$$\int_\gamma \sum c_n z^n\,dz = \sum \underbrace{\int_\gamma z^n\,dz}_{=0} = 0$$

Details: later.

Maximum principle: Suppose Ω is a bounded domain, f is continuous on $\overline{\Omega}$ and analytic on Ω.

Then $|f|$ attains its maximum value on the boundary. Furthermore, if $|f|$ has a local max inside Ω, then $f \equiv \text{const}$.



no place like this to "sit"

Key: Analytic fcns satisfy the averaging property:

$$\overline{D_r(a)} \subset \Omega: \quad f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$

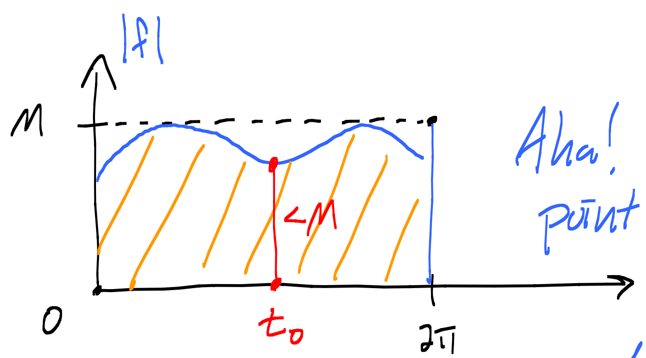
$$= \frac{1}{2\pi r}\int_0^{2\pi} f(a+re^{it})\,\underbrace{r\,dt}_{ds = r\,d\theta = \text{arc length}}$$

Key

Step 1: Special case of f analytic on bigger disc than $D_r(a)$, $|f|$ assumes its max M on $\overline{D_r(a)}$ at center a. Then $f \equiv \text{const}$.

$$M = |f(a)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(a+re^{it})\,dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(a+re^{it})|}_{\le M}\,dt$$

Hmmm.



Aha! If $|f| < M$ at any point t_0, then area $< M \cdot 2\pi$ and RHS $< M$.

Get $M < M$ ↯. $|f|$ can never dip below M on $C_r(a)$. Aha! Can let r slide from $0 < r < R$. See $|f| \equiv M$ on $D_R(a)$.

Cool thing: If $|f|$ is constant on $D_R(a)$, then $f \equiv$ const. (f analytic)

Why: $f(x+iy) = u(x,y) + i\,v(x,y)$

$$|f|^2 = c, \text{ a const.}$$

$$u^2 + v^2 = c \quad (*)$$

$$\begin{aligned} \frac{\partial}{\partial x}(*):&\ \cancel{2}u\,u_x + \cancel{2}v\,v_x = 0 \\ \frac{\partial}{\partial y}(*):&\ \cancel{2}u\,\underset{}{\overset{-v_x}{\overset{\shortparallel}{u_y}}} + \cancel{2}v\,\overset{u_x}{\overset{\shortparallel}{v_y}} = 0 \end{aligned}$$

C-R Equs

$$\boxed{\begin{array}{l} A\vec{x} = \vec{0} \\ \det A \neq 0 \Rightarrow \vec{x} = \vec{0}. \end{array}}$$

$$\underbrace{\begin{bmatrix} u_x & v_x \\ -v_x & u_x \end{bmatrix}}_{\text{det must } = 0} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

non-zero vector if $c \neq 0$

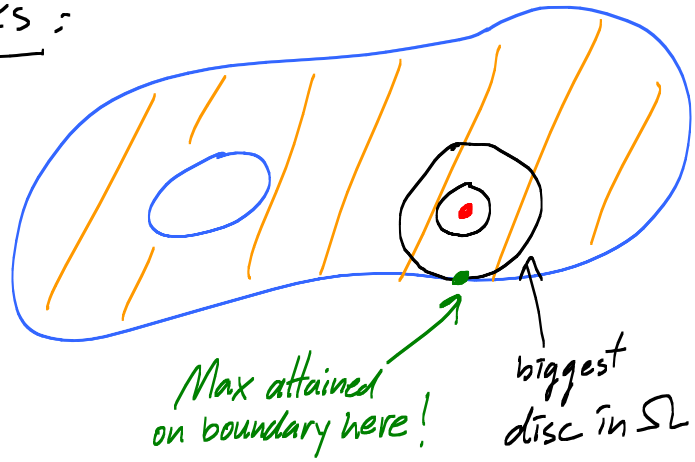
$$0 = \det = (u_x)^2 + (v_x)^2 = |f'|^2$$

by red box #1 $\boxed{f' = u_x + i\,v_x}$

Aha! $f' \equiv 0$, so $f \equiv$ const.

Fact: $|f|$ const on a domain $\Rightarrow$ $f \equiv$ const there.

Why Max Princ Works:



Ω bounded

$\bullet$ = Max of $|f|$ on $\overline{\Omega}$

Disc case $\Rightarrow$ $f \equiv$ const of modulus M on biggest disc.

Big fact: If $f \equiv$ const on $D_r(a) \subset \Omega$, then $f \equiv$ same const on whole domain Ω. (f analytic on Ω)

Harmonic fcns: $f = u + iv$ ← start with u. Get harm conj v.

f: analytic; u, v: harmonic

$$f(a) = u(a) + iv(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt + i \cdots$$

Aha! u satisfies Ave prop too.

Max Princ holds for harmonic fcns! (Ave prop $\Rightarrow$ Max princ)

Cool thing:
$$e^f = e^{u+iv} = e^u e^{iv}$$
$$|e^f| = |e^u|\underbrace{|e^{iv}|}_{1} = e^u$$

← Aha! local max of u gives local max of e^f $\Rightarrow$ $e^f \equiv c$, const

$|c| \equiv |e^f| \equiv e^u$ So $u \equiv \ln|c|$ is const.

This trick shows that (Max princ for analytic fcns)

$\Longrightarrow$ (Max prin for harmonic).

Exam 1 covers Lessons 1–15 on syllabus.

4.7 not on exam, but 4.6 might be.

Advice: Do problems for Lesson 15 before exam 1 on Mon.