Lesson 21 Power series

Lessons 20, 21 due Wed

1) $\mathbb{C}$ is an Algebraically closed field because every poly equ $P(z) = 0$ has a root in $\mathbb{C}$.

↑ coeff in $\mathbb{C}$

Note: $\mathbb{R}$ is not complete fied. Neither is $\mathbb{Q}$ ← rational #s.

2) $\mathbb{R}$, $\mathbb{C}$ are complete metric spaces, meaning:

Every Cauchy sequence in the set has a limit in the set.

Note: $\mathbb{Q}$ is not a complete metric space.

Def^n: A seq $\{z_n\}_{n=1}^{\infty}$ is a Cauchy seq if, for any $\varepsilon > 0$, there is an N such that

$$|z_n - z_m| < \varepsilon \quad \text{when} \quad n, m > N.$$

Completeness of $\mathbb{R}$ ($\Longrightarrow$ completeness of $\mathbb{C}$) is equivalent to "The least upper bound principle":

Every bounded set in $\mathbb{R}$ has a least upper bound.

Ex: $S = \{ r \in \mathbb{Q} : r^2 < 2 \}$.

$$\underbrace{\text{l.u.b.}}\ S = \sqrt{2} \notin \mathbb{Q}$$

↑ same as the supremum: $\sup S$

Hadamard's formula: Power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

has a Radius of Convergence (RoC) R given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

$R = 0$ or ∞ is allowed

Def^n: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} =$

$$\lim_{N\to\infty} \underbrace{\sup\{\sqrt[n]{|a_n|} : n \geq N\}}$$

can only go down as N goes up

This is a non-increasing seq of real numbers ≥ 0. So it has a limit.

Properties of RoC: Series converges absolutely inside $D_R(z_0)$. It diverges if $|z-z_0| > R$. Furthermore, series converges uniformly on any $D_r(z_0)$ with $0 < r < R$.

Morera's Thm $\Longrightarrow$ power series converges to an analytic fcn in $D_r(z_0)$, $r < R$. Let $r \nearrow R$ to be able to say p.s is analytic on $D_R(z_0)$.

<u>Note</u>: Convergence can be very tricky on the circle $C_R(z_0)$. Anything can happen!

<u>Easy way to understand why power series have a RoC.</u>

If $\sum_0^\infty a_n w^n$ converges at w, then we know that

$a_n w^n \to 0$ as $n \to \infty$. Take $\varepsilon = 1$.

Then $\exists\ N$ such $|a_n w^n| < 1$ ← ε

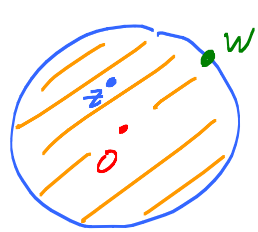
when $n > N$. If $|z| < |w|$, then

$$|a_n z^n| = |a_n z^n| \left|\frac{w^n}{w^n}\right| = \underbrace{|a_n w^n|}_{\substack{<1 \\ \text{if } n>N}} \left|\frac{z^n}{w^n}\right|$$

$$< \underbrace{\left|\frac{z}{w}\right|}_{<1 \text{ if } |z|<|w|.}^{\,n} \quad \text{when } n > N.$$

Aha! Can compare tail end of $\sum a_n z^n$ to a convergent geometric series!

<u>Conclude that</u> $\sum a_n z^n$ converges absolutely on $D_{|w|}(0)$.



<u>Notice</u>: Actually showed that if $|a_n r^n| < M$ for $n > N$, then $\sum a_n z^n$ converges for $|z| < r$.

<u>And</u>, notice that if $|a_n r^n|$ cannot be bounded, they don't $\to 0$ as $n \to \infty$, and therefore the series <u>diverges</u>.

<u>Fact</u>: $R = \sup \{ r : |a_n r^n| \text{ is a bounded seq} \}$

<u>EX</u>: $e^z = \sum_{n=0}^{\infty} \underbrace{\frac{z^n}{n!}}_{u_n}$

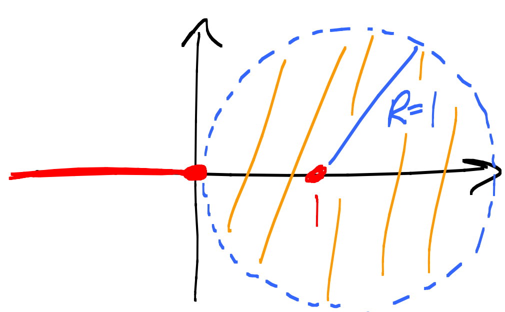
Ratio test: $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \frac{|z|}{n+1} \xrightarrow[\text{as } n \to \infty]{} 0$ $\begin{matrix} < 1 \text{ conv.} \\ > 1 \text{ div} \end{matrix}$

$L = 0 < 1$. Converges for <u>all</u> z. $RoC = \infty$.

<u>EX</u>: $\sum_{n=0}^{\infty} z^{n!}$ $\qquad R = 1$

<u>EX</u>: $f(z) = \operatorname{Log} z$

$$= \sum_{n=0}^{\infty} a_n (z-1)^n$$



where $a_n = \frac{f^{(n)}(1)}{n!}$

$$f(z) = \operatorname{Log} z \qquad f(1) = \operatorname{Log} 1 = \operatorname{Ln}|1| + i\cdot 0 = 0$$

$$f'(z) = \frac{1}{z} \qquad f'(1) = 1$$

$$f''(z) = \frac{-1}{z^2} \qquad f''(1) = -1$$

$$f'''(z) = \frac{2}{z^3} \qquad f'''(1) = 2$$

$$\vdots$$

$$f^{(n)}(1) = (n-1)!\,(-1)^{n+1}$$

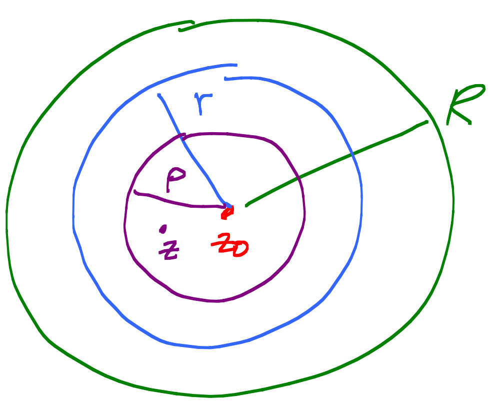
So $$a_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}}{n}$$

$$\operatorname{Log} z = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 - \tfrac{1}{4}(z-1)^4 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$$

Next time: 5.4 Hadamard's formula.

Only fact we haven't shown is that power series converge uniformly in $D_r(z_0)$ when $0 < r < R$.



need $0 < \rho < r < R$

Great behavior in $D_\rho(z_0)$