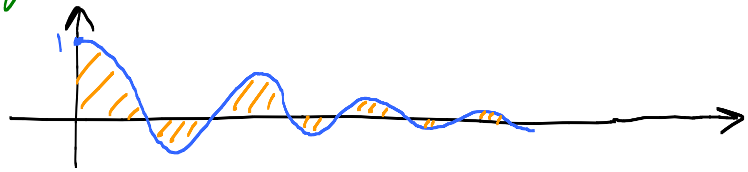
Lesson 29 Principal value integrals

28, 29, 30 due Wed

EX: $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$



Alternating series test shows that $\displaystyle\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\,dx$ exists.

But Sum (+areas) $= \infty$. Not absolutely integrable.

EX: $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$ is absolutely integrable

$$\left|\frac{\cos x}{x^2+1}\right| \le \frac{1}{x^2+1} \longleftarrow \int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \tan^{-1}x\Big|_{-R}^{R}$$

$$\to \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Consequence: $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}\,dx = \underbrace{\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{x^2+1}\,dx}_{\text{Principal value}} = \lim_{A\to-\infty}\int_A^0 + \lim_{B\to\infty}\int_0^B$

Fact: For abs integrable fcns : Princ Val = Value.

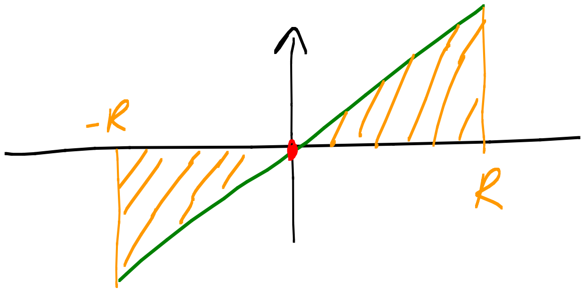
EX: $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$

Big x : $\sim \dfrac{1}{x^{3/2}} \longleftarrow$ integrable as $x\to\infty$

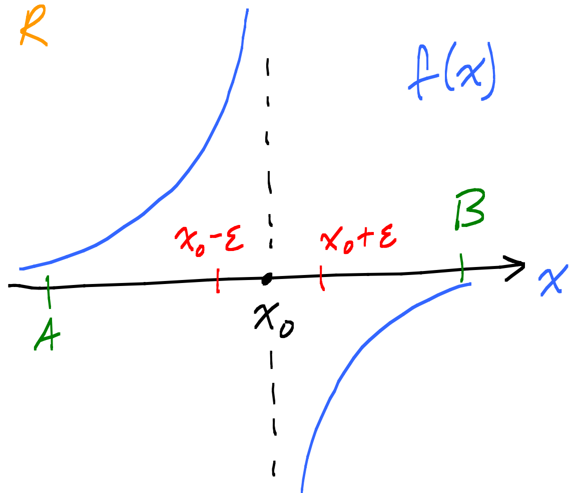
x near 0: $\sim \dfrac{1}{x^{1/2}} \longleftarrow$ integrable as $x\to 0$

OK to let $\varepsilon\to 0$, then $R\to\infty$.

Ex: $\text{P.V.}\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = 0$

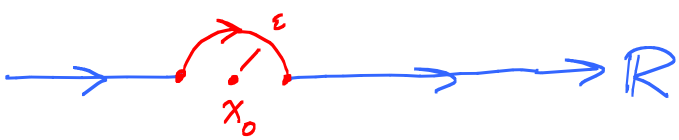


Another kind of P.V. $\int$:



$$\text{PV}\int_A^B f\,dx = \lim_{\varepsilon\to 0}\left(\int_A^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{B}\right) f\,dx$$

Why care :



Two important facts :

1) Jordan's lemma yields

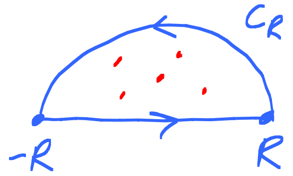
$$\text{PV}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}e^{ix}\,dx = 2\pi i \sum_{z_k\in\mathfrak{U}} \text{Res}_{z_k}\left(\frac{P(z)}{Q(z)}e^{iz}\right)$$

where $\mathfrak{U}$ is the set of zeroes of Q in the UHP.

Q non-vanishing on $\mathbb{R}$ and

$\deg Q \geq \deg P + 1$ ← improved from $+2$ without Jordan's.

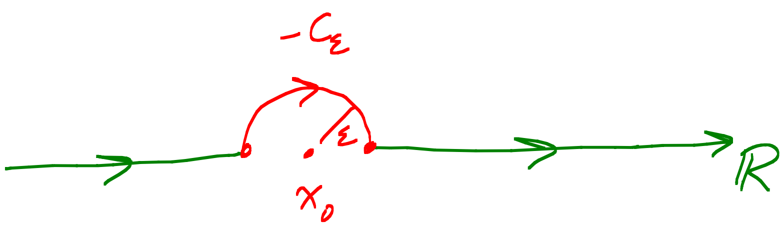
Why:



$$\left| \int_{C_R} \frac{P}{Q} e^{iz}\, dz \right| \longrightarrow 0 \quad \text{as } R \to \infty$$

use Basic poly estimate plus Jordan lemma estimate,

2) General fact: If $f(z)$ has a simple pole at $x_0 \in \mathbb{R}$:

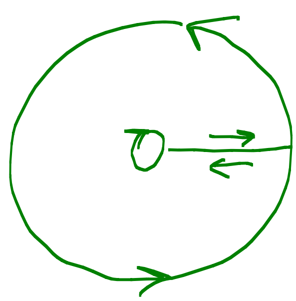


then
$$\lim_{\varepsilon \to 0} \int_{-C_\varepsilon} f\, dz = -\frac{1}{2} (2\pi i) \operatorname{Res}_{x_0} f$$

because clockwise

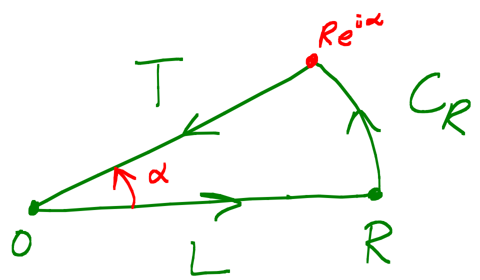
because half circle

Prob: $I = \displaystyle\int_0^\infty \frac{1}{x^5+1}\, dx$



Ouch! $\int$'s cancel!

Try



$f(z) = \dfrac{1}{z^5+1}$

L: $z(t) = t$, $0 \le t \le R$.
$z'(t) = 1$

$$\int_L f\, dz = \int_0^R \frac{1}{t^5+1}\, 1\, dt$$ want

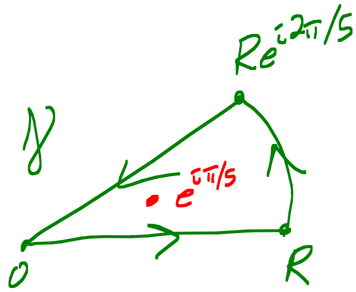
$$\left|\int_{C_R} f\,dz\right| \le \left(\max_{|z|=R} \frac{1}{|z^5+1|}\right) \text{Length}(C_R) \le \frac{1}{R^5-1}\cdot(\alpha R) \quad \text{if } R>1$$

$$\longrightarrow 0 \quad \text{as } R\to\infty.$$

Nice method: $-T$: $z(t) = t e^{i\alpha}$, $0 \le t \le R$ $\left[z'(t) = e^{i\alpha}\right]$

$$\int_T f\,dz = -\int_{-T} f\,dz = -\int_0^R \frac{1}{\underbrace{(te^{i\alpha})^5}_{\substack{\uparrow \\ \text{Aha! Need} \\ e^{i5\alpha}=1}}+1}\cdot \underbrace{e^{i\alpha}\,dt}_{z'(t)\,dt}$$

$e^{i5\alpha} = 1 \leftarrow$ Take $\alpha = \frac{2\pi}{5}$!



Now $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/5}} \frac{1}{z^5+1}$ $\quad \frac{1}{z^5+1}$ is $f(z)$, $z^5+1 \leftarrow g(z)$

$g(z) = z^5+1$ is zero at $z = e^{i\pi/5}$

$g'(z) = 5z^4$ is not zero at $e^{i\pi/5}$.

$$\left(\int_T + \int_L + \int_{C_R}\right) f\,dz = (2\pi i)\frac{f(e^{i\pi/5})}{g'(e^{i\pi/5})} = 2\pi i \frac{1}{5(e^{i\pi/5})^4}$$

$$-e^{i2\pi/5}\int_0^R \frac{1}{t^5+1}\,dt + \int_0^R \frac{1}{t^5+1}\,dt + \int_{C_R} f\,dz = \frac{2\pi i}{5e^{i4\pi/5}}$$

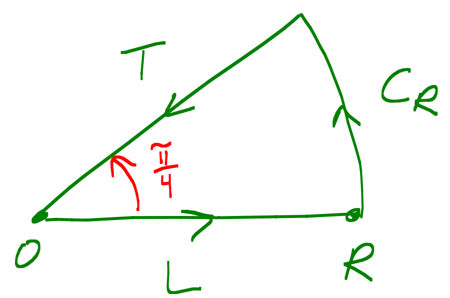
Now let $R\to\infty$: Get $\left(1-e^{i2\pi/5}\right) I + 0 = \dfrac{2\pi i}{5e^{i4\pi/5}}$

So $$I = \frac{2\pi i}{5e^{i4\pi/5}\left(1-e^{i2\pi/5}\right)} \cdot \frac{1}{\left(-e^{-i\pi/5}\right)\left(-e^{i\pi/5}\right)}$$

$$= \frac{\pi}{\underbrace{-5e^{i5\pi/5}}_{e^{i\pi}=-1}} \cdot \underbrace{\frac{2i}{e^{i\pi/5}-e^{-i\pi/5}}}_{1/\sin\frac{\pi}{5}} = \frac{\pi/5}{\sin \pi/5}$$

Fresnel integral: $\displaystyle\int_0^\infty \sin x^2\,dx = \lim_{R\to\infty}\int_0^R = \frac{\sqrt{2\pi}}{4}$

$f(z) = e^{-z^2}$

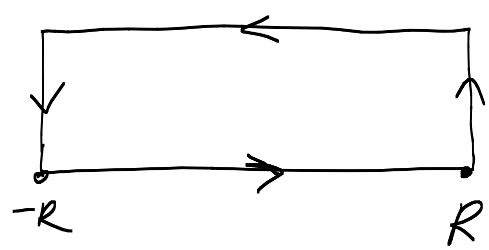


$\displaystyle\int_{C_R} \to 0$ by Jordan idea

Cauchy's Thm. $\displaystyle\int_\gamma = 0$

$\displaystyle\int_L$ want. $\displaystyle\int_T$ related to $\displaystyle\int_L$

EX:



Choosing γ is the hard part!