Lesson 35 Linear Fractional Transformations (LFT's)

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Book: LFTs = Möbius transformation

Me: Möbius transf is an LFT of form $\frac{z-a}{1-\bar{a}z}$

Note: If $ad-bc=0=\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}$,

then one row = multiple of other,

or one row might be zero : $L(z) \equiv$ const or not defined.

Basic maps:

1. $z \mapsto rz \quad (r>0)$ "Stretch"
2. $z \mapsto e^{i\theta} z$ "Rotation"
3. $z \mapsto z+b \quad (b \in \mathbb{C})$ "Translation"
4. $z \mapsto \frac{1}{z}$ "Inversion"

Note: 1, 2, 3 $\mathbb{C} \to \mathbb{C}$ 1-1 and onto

$\infty \mapsto \infty$ ← Because $\lim_{z\to\infty} L(z) = \infty$,

meaning that $\lim_{|z|\to\infty} |L(z)| = \infty$

Define $L(\infty) = \infty$ ← the "point at infinity"

4. $\mathbb{C}-\{0\} \xrightarrow[\text{onto}]{1-1} \mathbb{C}-\{0\}$ $\quad 0 \mapsto \infty$ Define $L(0)=\infty$

$\infty \mapsto 0$ Define $L(\infty) = 0$

$\frac{1}{z} : \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$

Fact: LFT's are 1-1 onto self maps of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Case: $L(z) = \frac{az+b}{cz+d} = Az+B$ if $c=0$ $\quad A = \frac{a}{d}$, $B = \frac{b}{d}$

$$\infty \mapsto \infty$$

Case $c \neq 0$: $L(z) = \dfrac{a + \frac{b}{z}}{c + \frac{d}{z}} \longrightarrow \dfrac{a}{c}$ as $z \to \infty$.

$$L(\infty) = \frac{a}{c}$$

L has a simple pole when $cz+d=0$, i.e., when $z = -\frac{d}{c}$

$$L\left(-\frac{d}{c}\right) = \infty$$

Inverse of $L(z)$: $w = \dfrac{az+b}{cz+d}$ $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$czw + dw = az + b$$

$$z(cw - a) = -dw + b$$

$$z = \frac{-dw + b}{cw - a} = \frac{dw - b}{-cw + a} \quad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

multiplying num & denom by $\frac{1}{\det}$ does not change an LFT.

Fact: $w = L(z) = \dfrac{az+b}{cz+d}$

$$z = L^{-1}(w) = \frac{dw - b}{-cw \quad a}$$

Cool fact: $L_1(L_2(z)) = \dfrac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} = \dfrac{Az + B}{Cz + D}$

where $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbb{A}$ "Group homomorphism" between LFT's and 2x2 matrices over $\mathbb{C}$ with det $\neq 0$.

$L^{-1} \sim \mathbb{A}^{-1}$

$L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2 \leftarrow$ matrix mult.

Fact: LFT's form a "group" under composition.

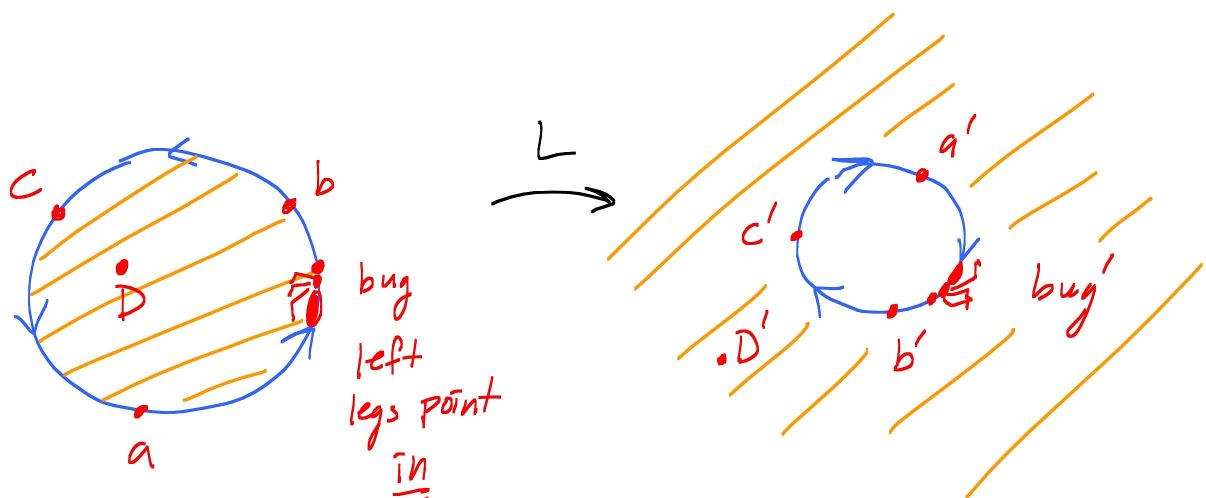
id : $L(z) = z = \dfrac{1 \cdot z + 0}{0 \cdot z + 1}$ ✓

Big facts: 1) LFTs : $\{\text{lines, circles}\} \longrightarrow \{\text{lines, circles}\}$

2) : $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ 1-1 onto

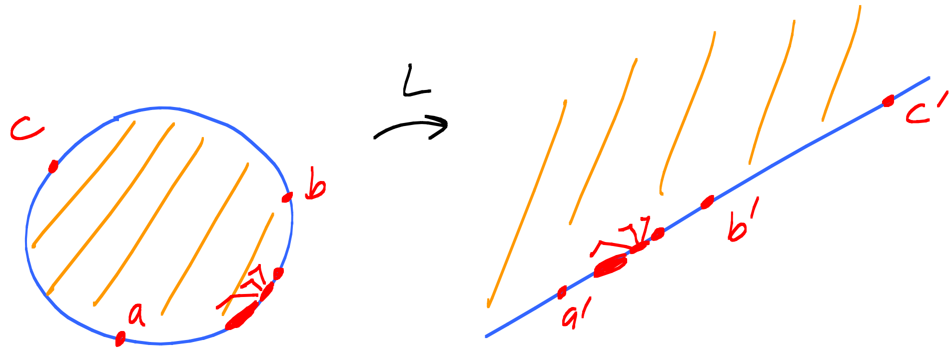
3) Analytic and $L'(z) \neq 0$, so conformal when mapping open sets in $\mathbb{C}$ to open sets in $\mathbb{C}$

EX:



left legs point in to image by conformality

Two tests: Bug test and "one point test"

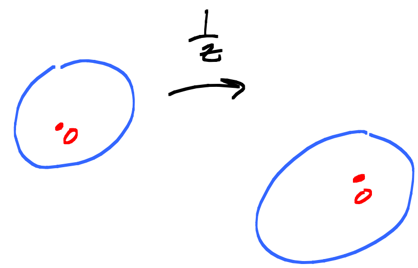


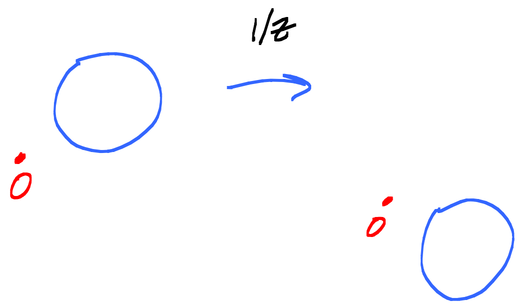
Why do LFTs take {lines, circles}

1, 2, 3 easy $\quad \{\text{lines}\} \to \{\text{lines}\}$

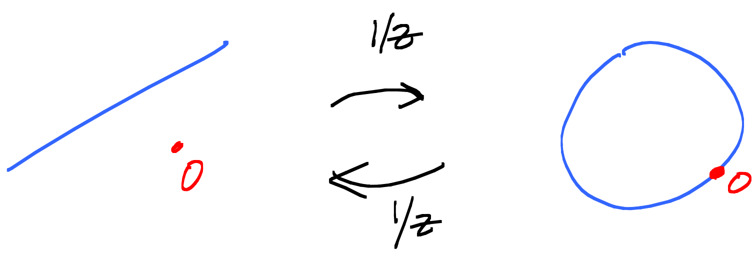
$\{\text{circles}\} \to \{\text{circles}\}$

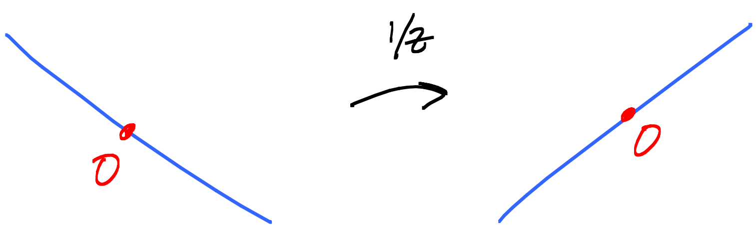
4 can jumble them up!





Proof: Analytic geometry





$z = te^{i\theta}$

$\frac{1}{z} = \frac{1}{t}e^{-i\theta}$

<u>Big fact</u>: LFTs are generated by 1–4 maps.

Why: $L(z) = \dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{\overbrace{\left(b - \frac{ad}{c}\right)}^{=Re^{i\psi}}}{\underset{c = re^{i\theta}}{c}z + d}$

$$\begin{array}{r} \frac{a}{c} \\ cz+d \,\big)\, \overline{az + b} \\ az + \frac{da}{c} \\ \hline b - \frac{da}{c} \end{array}$$

$c \neq 0$

L: Stretch by r, rotate by θ, translate by d, invert, stretch by R, rotate by ψ, translate by $\frac{a}{c}$.

Basic maps: {lines, circles} $\circlearrowleft$. Hence, so do LFTs.

Note: $c=0$ case. $L(z) = \underset{re^{i\theta}}{A}z + B$ ← no inversion