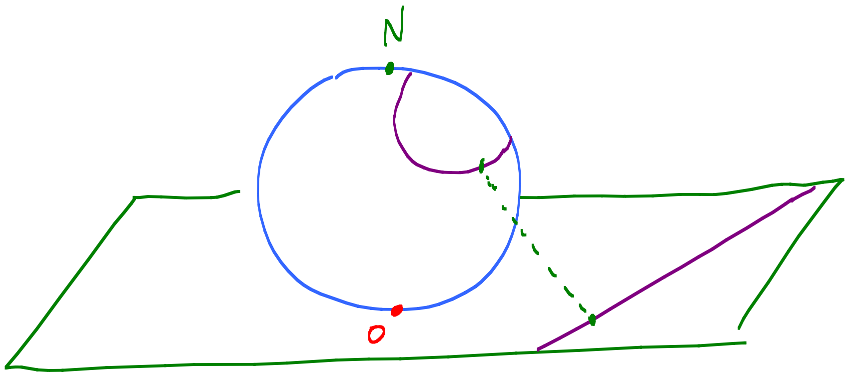
Lesson 36 LFTs

HWK 11: Lessons 31, 32, 35
due Friday, Nov. 22

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$



{lines, circles} in $\mathbb{C}$

$\sim$

{slices by planes} on Riemann sphere

Big fact A LFT that fixes 3 pts must be the identity.

Why: $L(z_j) = z_j \qquad j=1,2,3$

$$z_j = \frac{az_j+b}{cz_j+d}$$

$$c z_j^2 + (a-d) z_j - b = 0$$

Quadratic equ with 3 roots!

All coeff must $=0$.

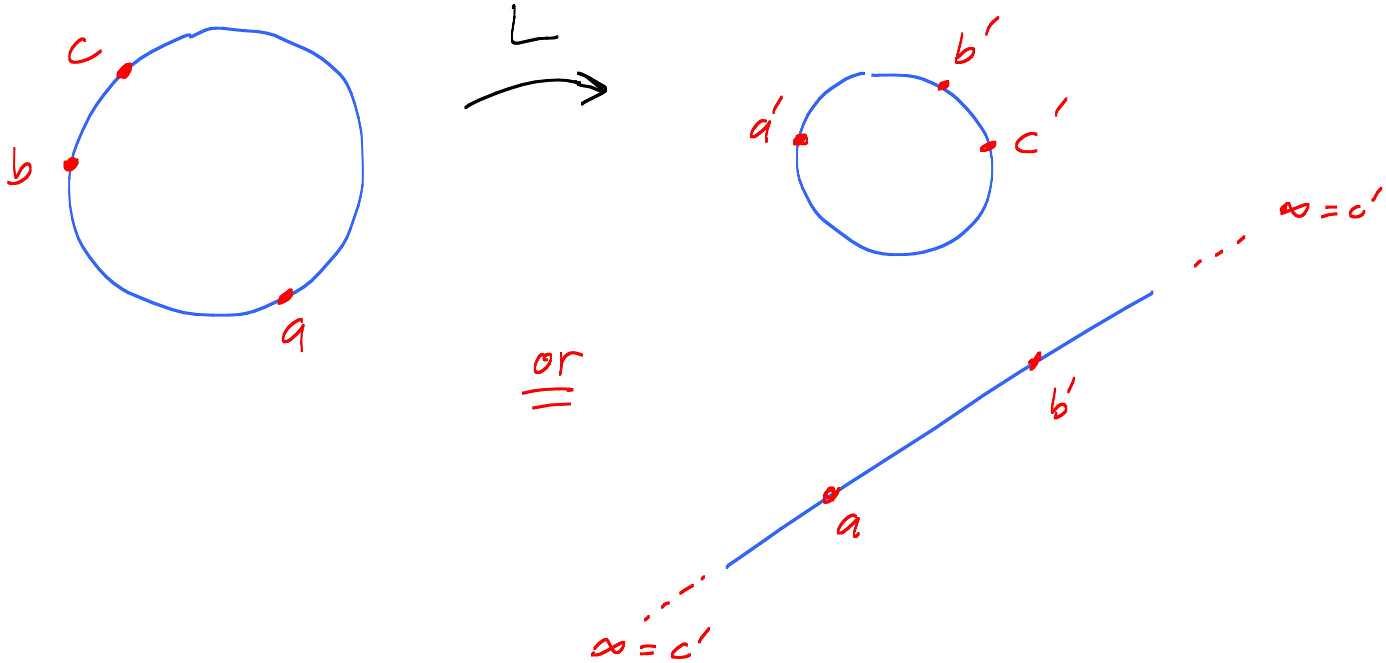
So $c=0, \quad a=d, \quad b=0$

$$L(z) = \frac{az+0}{0\cdot z + \cancel{d}_{a}} = z \checkmark$$

Note:
$ad-bc = ad \neq 0$
so $a \neq 0, d \neq 0$.

Consequence: 3 pts determine a circle. Enough to check $L(z_j)$, $j=1,2,3$ to figure out where a circle goes.

Think: Two points plus $\{\infty\}$ determine a line.



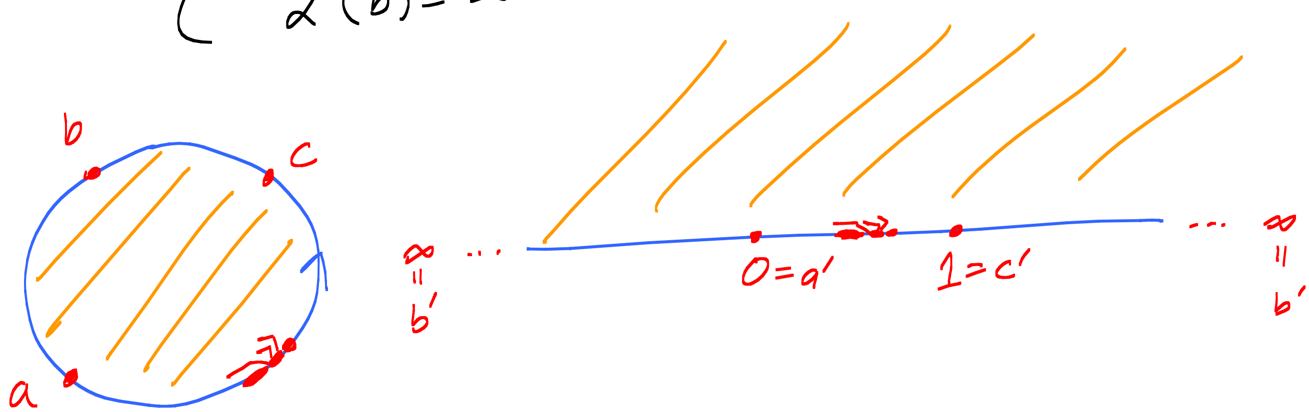
How to cook up LFTs:

Ex: $L(z) = \dfrac{z-a}{z-b}$ $\quad$ $a \longmapsto 0$, $\; b \longmapsto \infty$

Hmmm: What if I also want $L(c) = 1$?

Aha! $\quad \mathcal{L}(z) = \left(\dfrac{c-b}{c-a}\right) \cdot \dfrac{z-a}{z-b}$ does it!

$$\begin{cases} \mathcal{L}(a) = 0 \\ \mathcal{L}(c) = 1 \\ \mathcal{L}(b) = \infty \end{cases}$$



Prob: Find an LFT:

$$z_1 \mapsto w_1$$
$$z_2 \mapsto w_2$$
$$z_3 \mapsto w_3$$

z circle ↑   w circle ↑

Find L_1, L_2:

L_1: $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$

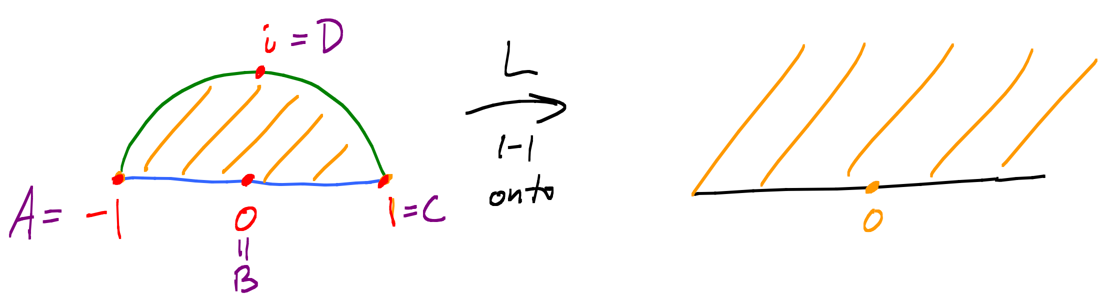
L_2: $w_1 \mapsto 0$, $w_2 \mapsto 1$, $w_3 \mapsto \infty$

$$L_1(z) = \frac{z_2 - z_3}{z_2 - z_1} \cdot \frac{z - z_1}{z - z_3}.$$

L_2: replace z's by w's.

Ans. $L_2^{-1}(L_1(z))$ does it. (Advice: Use matrix mult to find composition. Use inverse of a matrix to find L_2^{-1}.)

Prob: Find a conformal map



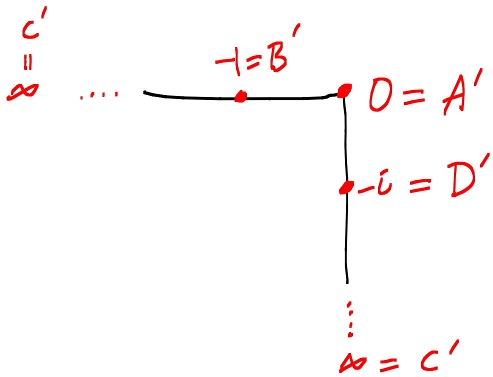
Big idea: Use $L(z) = \dfrac{z+1}{z-1}$ to blow $z=1$ out to ∞.

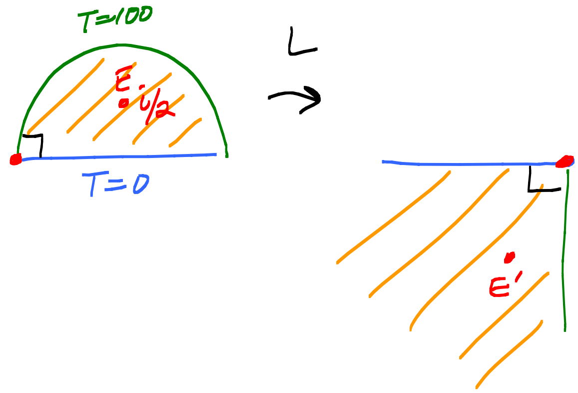
Then both boundary curves (line, circle) get mapped to lines.

Know $\begin{cases} L(1) = \infty & = C' \\ L(-1) = 0 & = A' \end{cases}$

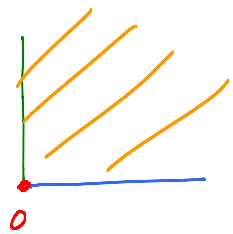
$$L(0) = \frac{0+1}{0-1} = -1 = B'$$

$$L(i) = \frac{i+1}{i-1} = -i = D'$$

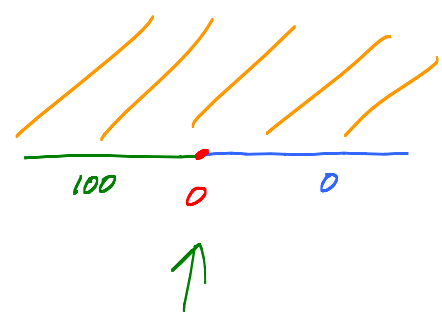




$e^{i\pi} z = -1 \cdot z$



z^2



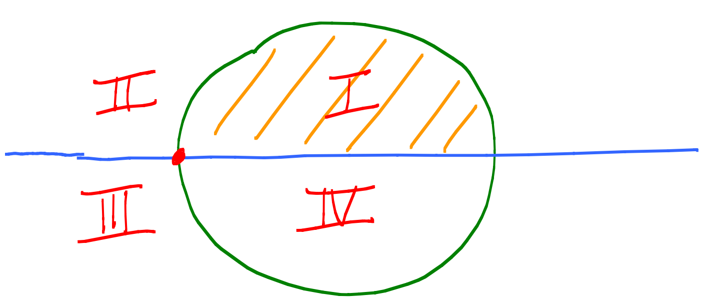
Solve heat prob by inspection:

$$\text{Temp} = u(z) = \frac{100}{\pi} \operatorname{Arg} z$$

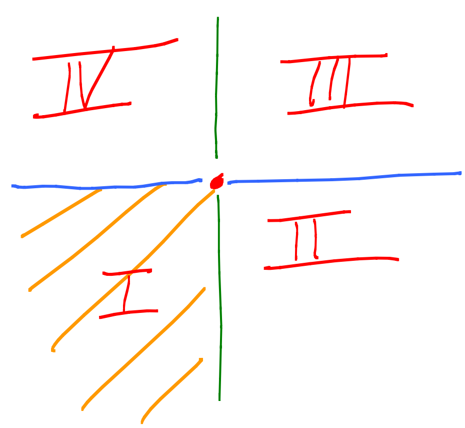
Sol^n on half disc: $\displaystyle \frac{100}{\pi} \operatorname{Arg}\left[\left(- \frac{z+1}{z-1} \right)^2 \right]$

$$= \frac{100}{\pi} \operatorname{Arg} \left(\frac{z+1}{z-1} \right)^2$$

$$= \frac{100}{\pi} \operatorname{Tan}^{-1} \left(\frac{\operatorname{Im} \left(\frac{(x+iy)+1}{(x+iy)-1} \right)^2}{\operatorname{Re} \left(\frac{(x+iy)+1}{(x+iy)-1} \right)^2} \right)$$



L



Conformal at •, $\hat{\mathbb{C}}$ ⟲ $\overset{1\text{-}1}{\text{onto}}$