MATH 425, Exam I

Each problem is worth 20 points.

1. Show that if $\varphi(x, y)$ is a twice continuously differentiable real valued *harmonic* function on a domain, then

$$\frac{\partial \varphi}{\partial x} - i\,\frac{\partial \varphi}{\partial y}$$

is analytic there.

2. Find a harmonic conjugate on $\mathbb{C}$ for

$$u(x, y) = x^2 - y^2 + xy + x + y.$$

3. Compute

$$\int_\gamma \frac{1}{z}\,dz$$

where γ is any curve in the plane that starts at $3 + 4i$ and ends at $-1 + i$ and that avoids the set $\{z : z = it, t \geq 0\}$ (i.e., the positive imaginary axis, including $z = 0$).

4. Define

$$I(a) = \int_C \frac{e^{5iz}}{(z-a)^4}\,dz,$$

where C is the unit circle parameterized in the counter clockwise direction and a is a complex number not on the unit circle. Compute $I(\frac{i}{3})$ and $I(3i)$. Is $I(a)$ an analytic function of a on the unit disc? Explain.

5. Show that

$$|e^{(z^2)}| \leq e^{|z^2|}$$

and identify conditions for equality to hold. Is it true that $|e^{(z^2)}|$ tends to infinity as $|z|$ tends to infinity?