Math 425

Exam 2

1. a) *(10 pts)* Suppose f has an isolated singularity at the origin and the limits $\lim_{t\to 0+} f(it)$ and $\lim_{t\to 0+} f(t)$ both exist, but are different. (t represents a real number here.) What must the type of the singularity of f be at $z = 0$? Explain.

b) *(10 pts)* If g is analytic on the unit disc and $g(1/n) = 1/n^2$ for each positive integer $n > 1$, find a formula for g. Explain.

2. *(20 pts)* Use the Ratio Test to find the radius of convergence about $z = 0$ of the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} z^{3n}.$$

Hint: $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$.

3. *(20 pts)* Compute

$$\int_0^{\infty} \frac{1}{x^3 + 8}\, dx$$

by integrating $1/(z^3 + 8)$ around the contour that follows the real line from zero to R, then follows the circle Re^{it} from $t = 0$ to $t = 2\pi/3$, and then follows the line $te^{i2\pi/3}$ from $t = R$ back to $t = 0$. Use the Residue Theorem and let $R \to \infty$.

4. *(20 pts)* Convert the integral

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$$

into a contour integral of the form $\int_C f(z)\, dz$ where f is a rational function and C is the unit circle parameterized in the counterclockwise sense. Find f, then use the residue theorem to compute the integral.

5. *(20 pts)* Compute

a) $\operatorname{Res}_0 \dfrac{e^{2z}}{z^4}$

b) $\operatorname{Res}_2 \dfrac{e^{3z}}{(z^2 - 4)^2}$