Exam 2 solutions

1. a) If had a removable singularity at $z=0$, then $\lim_{z\to 0} f(z)$ would exist and both $\lim_{t\to 0+} f(t)$ and $\lim_{t\to 0+} f(it)$ would be the same. If f had a pole, then $\lim_{z\to 0} f(z)=\infty$, and the two limits would not exist. So f must have an essential singularity at $z=0$.

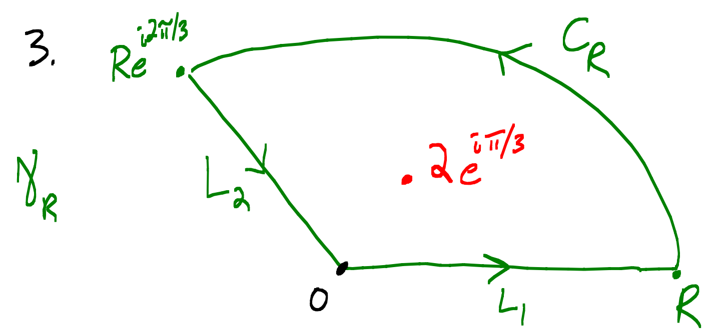
b) Let $f(z)=z^2$. Notice that $f(a_n)=g(a_n)$ where $a_n=\frac{1}{n}$, $n=2,3,\ldots$ is a sequence in $D_1(0)$ with a limit point at 0, which is a point in $D_1(0)$. The Identity Theorem yields that $f(z)\equiv g(z)$ on $D_1(0)$, i.e., $g(z)=z^2$.

2. Let $u_n=\frac{n^n}{3^n n!} z^{3n}$.

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\left[\frac{(n+1)^{(n+1)}}{3^{n+1}(n+1)!} z^{3(n+1)}\right]}{\left[\frac{n^n}{3^n n!} z^{3n}\right]}\right| = \frac{(n+1)^n}{3\, n^n}\,|z^3|$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^n |z|^3 \xrightarrow[n\to\infty]{} \frac{e}{3}|z|^3 \begin{array}{l} <1 \quad \text{Converges} \\ >1 \quad \text{Diverges}\end{array}$$

Ratio Test shows that $R=\sqrt[3]{\frac{3}{e}}$

3. 

Let $f(z)=\frac{1}{z^3+8}$

Step 1: $\left|\int_{C_R} f(z)\,dz\right| \le$

$\left(\max_{C_R}|f|\right)\cdot \text{Length}(C_R)$

$\le \frac{1}{R^3-8}\cdot\left(\frac{2\pi}{3}R\right)$ if $R>2$

$\longrightarrow 0$ as $R\to\infty$

Step 2: L_1 : $z(t)=t$, $0\le t\le R$

$$\int_{L_1} f\,dz = \int_0^R \frac{1}{t^3+8}\cdot 1\,dt \leftarrow \text{want}$$

Step 3: $-L_2$: $z(t) = t e^{i2\pi/3}$, $0 \le t \le R$

$z'(t) = e^{i2\pi/3}$

$$\int_{L_2} f\,dz = -\int_{-L_2} f\,dz = -\int_0^R \frac{1}{(te^{i2\pi/3})^3+8}\underbrace{\left[e^{i2\pi/3}\cdot dt\right]}_{dz = z'(t)\,dt}$$

$$= -e^{i2\pi/3}\int_0^R \frac{1}{t^3+8}\,dt \quad \leftarrow \text{want}$$

Step 4: Residue theorem :
$$\int_{\gamma_R} f(z)\,dz = 2\pi i \sum \operatorname{Res}_a f \quad \left(\begin{array}{c} a \text{ zero of denom} \\ \text{in } \gamma_R \end{array}\right)$$

$$\left(\underbrace{\int_{L_1} + \int_{L_2}} + \int_{C_R}\right) f\,dz = 2\pi i \operatorname{Res}_{2e^{i\pi/3}} \frac{1}{z^3+8} \leftarrow \text{simple zero of denom at } 2e^{i\pi/3}$$

$$(1-e^{i2\pi/3})\int_0^R \frac{1}{t^3+8}dt + \int_{C_R} f\,dz = 2\pi i \cdot \left.\frac{1}{3z^2}\right|_{z=2e^{i\pi/3}}$$

$R\to\infty$

$$(1-e^{i2\pi/3})\int_0^\infty \frac{1}{t^3+8}\,dt + 0 = \frac{2\pi i}{3\cdot 4e^{i2\pi/3}}$$

$$I = \underbrace{\frac{\pi}{12}\,\frac{2i}{e^{i2\pi/3}(1-e^{i2\pi/3})}}_{\text{ans}} \cdot \frac{1}{e^{-i\pi/3}\cdot e^{i\pi/3}}$$

$$= \frac{\pi}{12}\,\frac{2i}{\underbrace{e^{i3\pi/3}}_{e^{i\pi}=-1}\left(e^{-i\pi/3}-e^{i\pi/3}\right)} = \frac{\pi/12}{\sin \pi/3} = \frac{\pi/12}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{\pi\sqrt{3}}{18}$$

4. C : $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. $\quad z'(\theta) = ie^{i\theta}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z(t)+\frac{1}{z(t)}}{2}$$

$$I = \int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_0^{2\pi} \frac{1}{2 + \dfrac{z(\theta)+\frac{1}{z(\theta)}}{2}} \; \frac{\overbrace{i e^{i\theta}\, d\theta}^{dz}}{\underbrace{i e^{i\theta}}_{i z(\theta)}}$$

$$= \boxed{\int_C \frac{1}{2 + \dfrac{z+\frac{1}{z}}{2}} \cdot \frac{1}{iz}\, dz}$$

$$= \frac{2}{i}\int_C \frac{1}{z^2+4z+1}\, dz$$

Roots: $\dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

$\underbrace{-2+\sqrt{3}}$ inside. $-2-\sqrt{3}$ outside

simple zero

$f(z) = z^2 + 4z + 1$

$f'(z) = 2z + 4$

$$I = \frac{2}{i}\cdot 2\pi i \operatorname{Res}_{-2+\sqrt{3}} \frac{1}{z^2+4z+1} = 4\pi \cdot \frac{1}{2(-2+\sqrt{3})+4} = \frac{2\pi}{\sqrt{3}}$$

5. a) $$\frac{e^{2z}}{z^4} = \frac{1}{z^4}\left[1 + (2z) + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots\right]$$

$$= \frac{1}{z^4} + \frac{2}{z^3} + \frac{1}{z^2} + \frac{8/3!}{z} + \text{power series}$$

$$\operatorname{Res}_0 = \frac{8}{3\cdot 2\cdot 1} = \frac{4}{3}$$

b) $$\frac{e^{3z}}{(z-2)^2(z+2)^2} = \frac{1}{(z-2)^2}\left[\underbrace{\frac{e^{3z}}{(z+2)^2}}\right]$$

$H(z) = A_0 + A_1(z-2) + \cdots$

$$= \frac{A_0}{(z-2)^2} + \frac{A_1}{z-2} + \text{power series}.$$

$$\operatorname{Res}_2 = \frac{H'(2)}{1!} = \left.\frac{(3e^{3z})(z+2)^2 - e^{3z}\cdot 2(z+2)}{(z+2)^4}\right|_{z=2}$$

$$= \frac{3e^6\cdot 16 - e^6\cdot 8}{4^4} = \frac{5}{32}e^6$$