Math 425

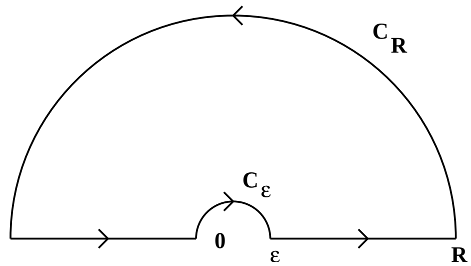
Exam 2

Each problem is worth 25 points

1. Use the contour pictured below to compute

$$\int_0^\infty \frac{\operatorname{Ln} x}{(x^2+4)^2}\,dx.$$

Define the branch of a complex logarithm that you use and justify your calculations and limits.



2. Assume that f is analytic on $D_1(0) - \{0\}$ and satisfies the estimate

$$|f(z)| \le \frac{C}{|z|^\alpha}$$

there for some constant $C > 0$ and constant α with $0 < \alpha < 1$. Prove that f has a removable singularity at $z = 0$. Hint: Consider the type of the singularity of $F(z) = zf(z)$ at the origin.

3. Find an analytic function that maps $\{z : 0 < \text{Re } z < 1\}$ one-to-one onto the first quadrant.

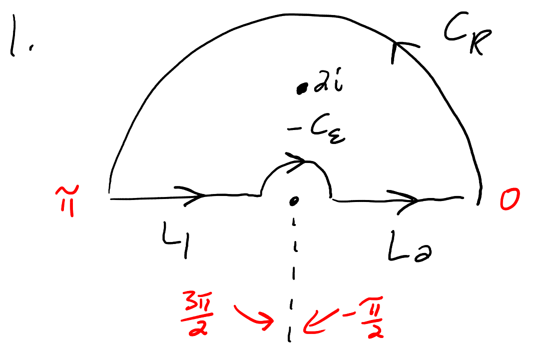
4. Prove that there are no polynomials of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

satisfying $|P(z)| < 1$ when $|z| = 1$. Hint: Rouché's

MA 425 Exam 2 solutions

1.



$\log z = \log_{-\frac{\pi}{2}} z = \operatorname{Ln}|z| + i\theta$

where $\theta \in \arg z$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

Note that $|\log z| \le |\operatorname{Ln}|z|| + |\theta|$
$< |\operatorname{Ln}|z|| + \frac{3\pi}{2}$

$0 < \varepsilon < 1 < 2 < R.$ On C_R: $\displaystyle\left|\int_{C_R} \frac{\log z}{(z^2+4)^2}\, dz\right| \le \max_{z \in C_R} \left|\frac{\log z}{(z^2+4)^2}\right| (\pi R)$

$|\log Re^{it}| = |\operatorname{Ln} R + it|$
$\le \operatorname{Ln} R + t \le \operatorname{Ln} R + \pi$

$\le \dfrac{\operatorname{Ln} R + \pi}{(R^2-4)^2} \cdot (\pi R)$ if $R > 2$

$\longrightarrow 0$ as $R \to \infty$.

On C_ε: $\displaystyle\left|\int_{C_\varepsilon}\right| \le \frac{|\operatorname{Ln}\varepsilon| + \pi}{(4-\varepsilon^2)^2} \cdot (\pi\varepsilon)$ if $0 < \varepsilon < 1$

$\longrightarrow 0$ as $\varepsilon \to 0$ (since $\varepsilon \operatorname{Ln} \varepsilon \to 0$ as $\varepsilon \to 0$)

$$\int_{L_2} \frac{\log z}{(z^2+4)^2}\, dz = \int_\varepsilon^R \frac{\operatorname{Ln} t + i\cdot 0}{(t^2+4)^2}\, 1 \cdot dt = I_\varepsilon^R \leftarrow \text{want}$$

$-L_1$: $z(t) = -t$, $\varepsilon \le t \le R$. $z'(t) = -1$

$$\int_{L_1} \frac{\log z}{(z^2+4)^2}\, dz = -\int_{-L_1} = -\int_\varepsilon^R \frac{\operatorname{Ln}|-t| + i\pi}{((-t)^2+4)^2} (-1)\, dt$$

$$= \underbrace{\int_\varepsilon^R \frac{\operatorname{Ln} t}{(t^2+4)^2}\, dt}_{I_\varepsilon^R} + \underbrace{i\pi \int_\varepsilon^R \frac{1}{(t^2+4)^2}\, dt}_{\text{don't care what this is.}}$$

$$\frac{\log z}{(z^2+4)^2} = \frac{1}{(z-2i)^2} \Bigg[\underbrace{\frac{\log z}{(z+2i)^2}}_{A_0 + A_1(z-2i) + \cdots} \Bigg] = \frac{A_0}{(z-2i)^2} + \frac{A_1}{z-2i} + \cdots$$

$\leftarrow \operatorname{Res}_{2i}$

$$\operatorname{Res}_{2i} \frac{\log z}{(z^2+4)^2} = A_1 = \frac{1}{1!}\left[\frac{\log z}{(z+2i)^2}\right]'\Bigg|_{z=2i}$$

$$= \frac{\left(\frac{1}{z}\right)(z+2i)^2 - (\log z)\left[2(z+2i)\right]}{\left[(z+2i)^2\right]^2}\Bigg|_{z=2i}$$

$$= \frac{\left(\frac{1}{2i}\right)(4i)^2 - \left(\ln 2 + i\frac{\pi}{2}\right)2\cdot 4i}{(4i)^4} = \frac{8i - (8\ln 2)i + 4\pi}{256}$$

$$= \frac{\pi}{64} + \frac{1-\ln 2}{32}i$$

Finally, $\left(\int_{L_1} + \int_{L_2} + \int_{C_\varepsilon} + \int_{C_R}\right) \frac{\log z}{(z^2+4)^2}\,dz = 2\pi i \operatorname{Res}_{2i} \frac{\log z}{(z^2+4)^2}$

$\varepsilon \to 0$

$R \to \infty$

$$2I + i\pi\int_0^\infty \frac{1}{(t^2+4)^2}dt + 0 + 0 = 2\pi i\left(\frac{\pi}{64} + \frac{1-\ln 2}{32}i\right)$$

$$I = \frac{1}{2}\operatorname{Re}\left[RHS\right] = \frac{\pi(\ln 2 - 1)}{32}$$

2. Let $F(z) = zf(z)$. Note that $|F(z)| = |z||f(z)| \leq C|z|^{1-\alpha}$

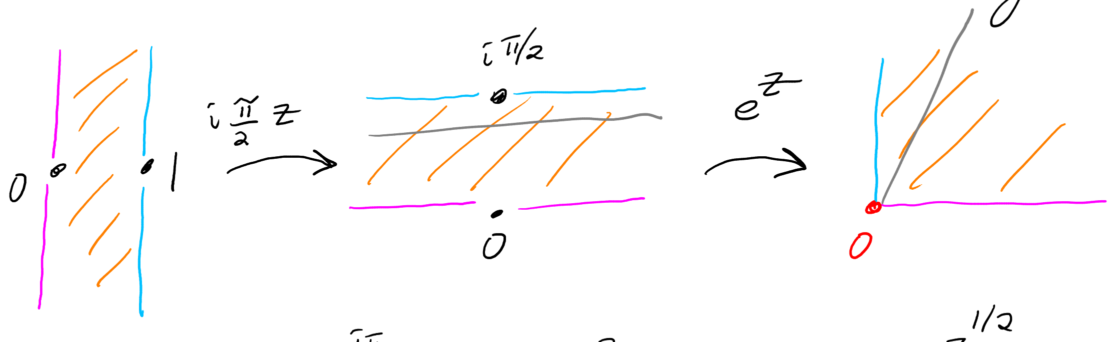
$|f(z)| \leq \frac{C}{|z|^\alpha}$

$0<\alpha<1$

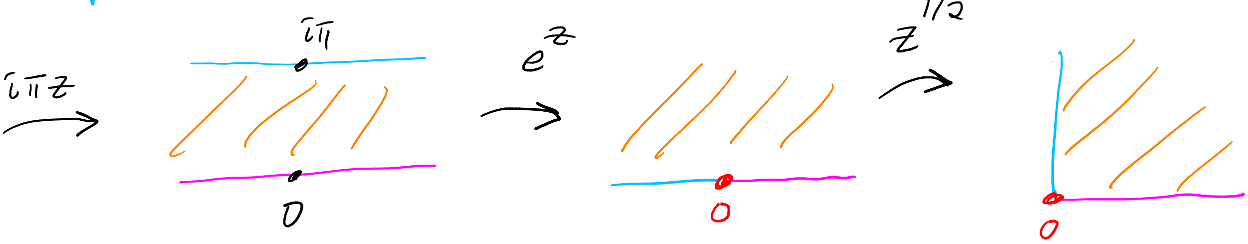
and this $\to 0$ as $z \to 0$ since $1-\alpha > 0$ when $0<\alpha<1$. Hence F has a removable singularity at $z=0$. Furthermore, if we define $F(0) = \lim_{z\to 0} F(z) = 0$, F is analytic on $D_1(0)$ and has a zero of order $m \geq 1$. Hence $F(z) = z^m G(z)$ where G is analytic on $D_1(0)$.

Finally, $f(z) = \frac{F(z)}{z} = \underbrace{z^{m-1} G(z)}_{\text{analytic on } D_1(0)}$ if $z \neq 0$

and we see that f has a removable singularity at $z=0$.

3. 

$f(z) = e^{i\frac{\pi}{2}z}$

or 

4. Let $f(z) = -z^n$ and $h(z) = P(z)$.

$\boxed{|P(z)| < 1 \text{ when } |z| = 1}$ would imply that

$$|h(z)| = |P(z)| < \underbrace{|-z^n|}_{=1} \quad \text{when } |z| = 1.$$

$h(z) = P(z)$ $\qquad$ $f(z) = -z^n$

$f(z) = ?$ $\quad f+h$?

Rouché's $\Rightarrow$ f and $f+h$ have same # zeroes in $D_1(0)$. $f(z) = -z^n$ has <u>n zeroes.</u>

$f(z) + h(z) = \sum_{n=0}^{n-1} a_k z^k$ is a polynomial of deg $\leq n-1$ and is either $\equiv 0$ or has at most $n-1$ zeroes.

Contradiction. Hence, it cannot happen that $|P(z)| < 1$ when $|z| = 1$.

Rouché: $\begin{cases} |f-g| < |f| \text{ on } \gamma \\ f, g \text{ same \# zeroes in } \gamma \end{cases}$

in some books