Math 428

Homework 7

1. Assume that $f(x)$ is a real valued C^1-smooth function on the whole real line such that $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite. Show that

$$\alpha(w) = \int_{-\infty}^{\infty} f(x) \sin wx \; dx$$

goes to zero as $w \to \infty$. Hint: Let $\epsilon > 0$. Explain why there is a positive integer N such that the part of the integral defining $\alpha(w)$ outside $[-N, N]$ is less than $\epsilon/2$ in absolute value. Next, use integration by parts (like you did in a previous homework problem) to show that the part of the integral on $[-N, N]$ goes to zero as $w \to \infty$, and hence can be made less than $\epsilon/2$ as well.

2. Show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\; dx$ is finite by using the estimate, $e^{-x^2} < e^{-|x|}$ when $|x| > 1$.

3. Denote the value of the integral in problem 2 by c (which we have shown in class is equal to $\sqrt{\pi}$) and let $\phi(x) = (1/c)e^{-x^2}$. For $t > 0$, define $\phi_t(x) = (1/t)\phi(x/t)$. Notice that $\int_{-\infty}^{\infty} \phi(x)\; dx = 1$. Show that $\int_{-\infty}^{\infty} \phi_t(x)\; dx = 1$ too. Graph ϕ_t for $t = 1$ and $t = 1/2$ on a single graph. Given $\delta > 0$, show that

$$\int_{\delta}^{\infty} \phi_t(x)\; dx$$

tends to zero as $t \to 0$. (The same is true about the integral from $-\infty$ to $-\delta$.)

4. Using the same notation as in the previous problem, show that, if f is a bounded continuous function on the real line, then

$$\int_{-\infty}^{\infty} f(x)\phi_t(x)\; dx$$

tends to $f(0)$ as $t \to 0$. Hint: $\phi_t(x)$ has three key properties that make it act like a “good kernel.”

5. Prove Weyl’s Criterion, that given a sequence of angles θ_n, the points $e^{i\theta_n}$ are equidistributed on the unit circle if and only if

$$\frac{1}{N}\sum_{n=1}^{N} e^{im\theta_n}$$

tends to zero as N tends to infinitey for each positive integer m.

Hints: The limit for a postive integer m implies the limit for negative integer $-m$ because one is the conjugate of the other. The $m = 0$ case is special (and different). Follow the proof of Weyl’s Theorem: true for Fourier basis functions implies true for trigonometric polynomials implies true for continuous 2π-periodic functions implies ...