Lecture 20 Fund. Thm. Algebra

Rules of diff'tion hold for complex fcns.

Rational fcns are complex diff'ble where denom $\neq 0$.

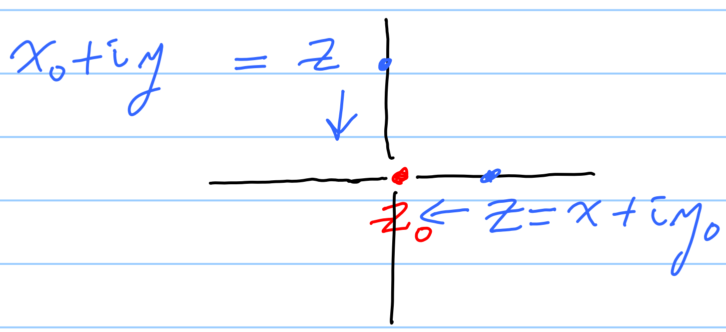
Fact: $f(x+iy) = u(x,y) + i\,v(x,y)$

If f is complex diff'ble, then u and v satisfy the Cauchy-Riemann Eqns:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

EX: $f(x+iy) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u} + i\,\underbrace{2xy}_{v}$

$f(z_0) = w_0 \qquad z_0 = x_0 + iy_0$



$$\text{Horiz} = \frac{f(x+iy_0) - f(x_0+iy_0)}{(x+iy_0)-(x_0+iy_0)} = \frac{[u(x,y_0)-u(x_0,y_0)] + i[v(x,y_0)-v(x_0,y_0)]}{x-x_0}$$

$$= \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\,\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$

$$\boxed{DQ \longrightarrow \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$$

<u>Vertical</u>: $\dfrac{f(x_0+iy)-f(x_0+iy_0)}{(x_0+iy)-(x_0+iy_0)}$

$$\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \frac{i\,[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)}$$

$$\boxed{DQ \longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$$

$$-i = \frac{1}{i}$$

Equate red boxes to get C-R eqns.

<u>Fact</u>: $R(z) = \dfrac{P(z)}{Q(z)}$, P, Q polys.

R analytic, so C-R eqns hold.

<u>Claim</u>: $\underbrace{\operatorname{Re} R(z)}_{u}$ and $\underbrace{\operatorname{Im} R(z)}_{v}$ are <u>harmonic</u>!

<u>Step 1</u>: u and v are C^∞ smooth where

$Q \neq 0$.

<u>Step 2</u>: C-R Eqns

$$u_x = v_y \qquad (A)$$
$$u_y = -v_x \qquad (B)$$

$\frac{\partial}{\partial x}(A)$: $\quad u_{xx} = v_{xy}$

$\frac{\partial}{\partial y}(B)$: $\quad u_{yy} = -v_{yx}$

($v_{xy} = v_{yx}$ if v C^2-smooth

So $\quad u_{xx} = -u_{yy}$. Aha! $\Delta u \equiv 0$.

Similarly, v is harmonic.

<u>Pf</u>: Fund. Thm. Alg. $\quad$ (Deg $P \geq 1$)

Suppose $P(z)$ is a complex poly with <u>no zeroes</u> in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is analytic on $\mathbb{C}$.

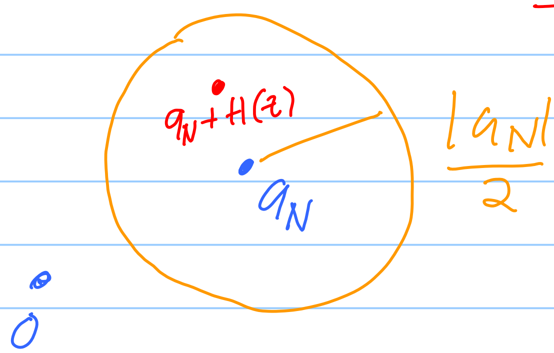
<u>Claim</u>: $\frac{1}{P(z)}$ is <u>bounded</u> on $\mathbb{C}$.

$$P(z) = a_N z^N + \cdots + a_1 z + a_0$$

$a_N \neq 0$, $N \geq 1$.

$$= z^N \Big(a_N + \underbrace{\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}}_{H(z)} \Big)$$

$\to 0$ as $|z| \to \infty$.

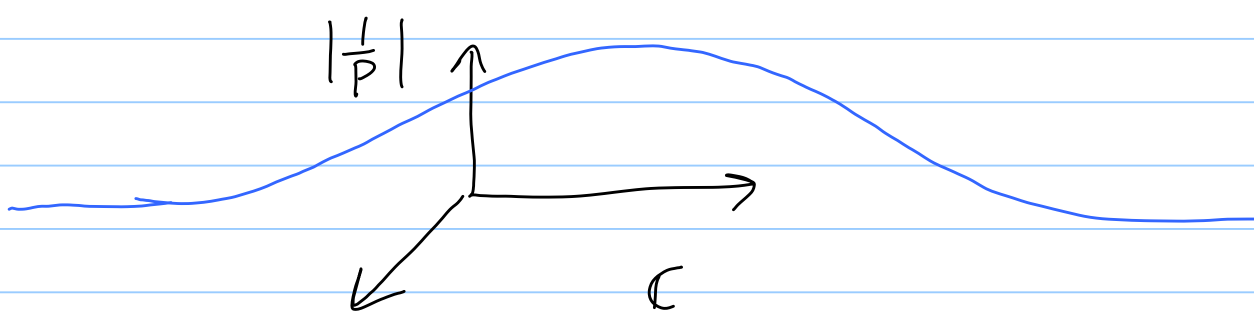


There is an $R > 0$ such that $|a_N + H(z)| > \frac{|a_N|}{2}$ if $|z| > R$.

So $|P(z)| = |z^N|\,|a_N + H(z)| > C|z|^N$

where $C = \frac{|a_N|}{2}$ when $|z| > R$.

Now, $\lim\limits_{|z| \to \infty} \left| \frac{1}{P(z)} \right| = 0$.



Claim: $\left|\frac{1}{P}\right|$ is bounded on $\mathbb{C}$.

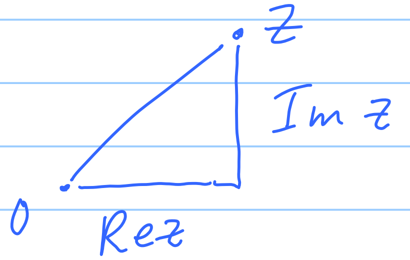
$\exists\, R>0$ such that $\left|\frac{1}{P(z)}\right| < 1$ if $|z|>R$.

But $\left|\frac{1}{P(z)}\right|$ is continuous on $\mathbb{C}$, so on $\overline{D_R(0)}$

closed, bounded

So $\left|\frac{1}{P(z)}\right|$ assumes its max M on $\overline{D_R(0)}$.

Now $\max(M,1)$ is a bound on $\mathbb{C}$.



$|\operatorname{Re} z| \le |z|$

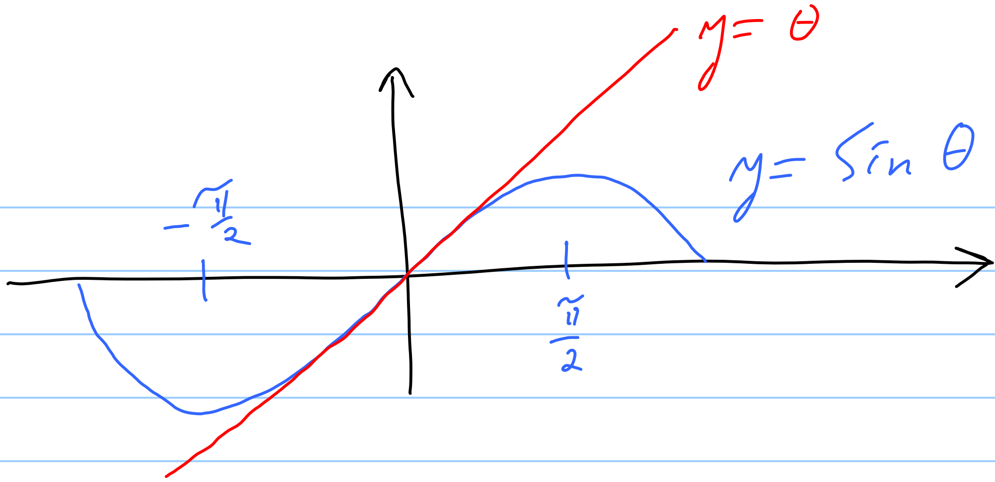
$|\operatorname{Im} z| \le |z|$

Aha! $\begin{cases} u = \operatorname{Re} \frac{1}{P(z)} \\ v = \operatorname{Im} \frac{1}{P(z)} \end{cases}$ are bounded harmonic fcns on $\mathbb{C}$

Liouville's $\Longrightarrow$ they are constant.

$\Longrightarrow P(z)$ is constant $\lightning$.

$\left[\ \frac{d^N}{dz^N} P(z) = N!\, a_N \neq 0.\ \right]$



$$|\sin\theta| < |\theta| \quad \text{for all } \theta$$

$$\text{MVT}: \quad \frac{\sin\theta - \sin 0}{\theta - 0} = \cos\alpha \qquad 0<\alpha<\theta$$

$$\int_{-\pi}^{\pi} \underbrace{\left|\frac{\sin(N+\frac{1}{2})\theta}{\theta}\right|}_{\text{even}} d\theta = 2\int_0^{\pi}$$

$$\sim \int_{\pi}^{N\pi} \left|\frac{\sin\psi}{\psi}\right| d\psi \qquad ???$$

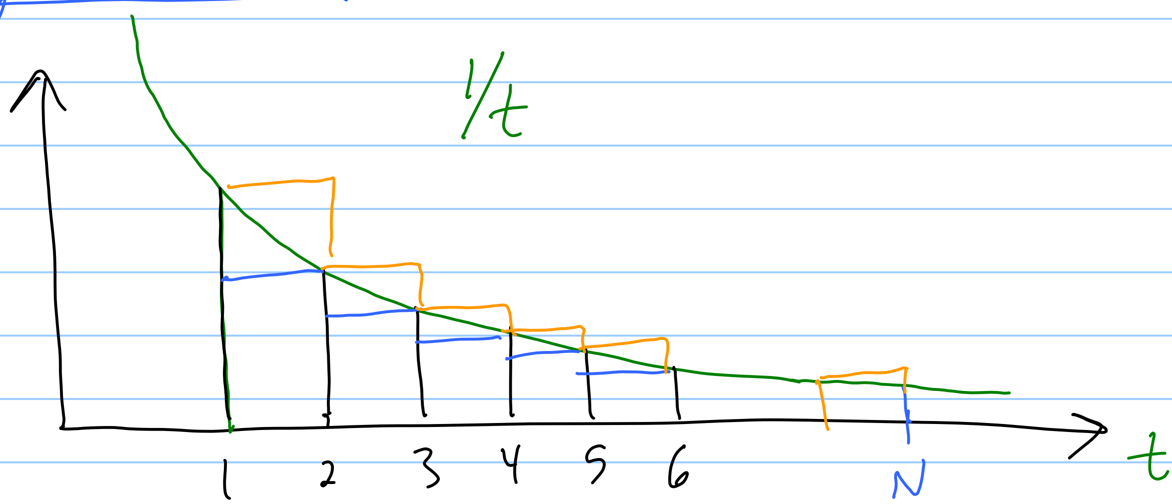
$$\psi = (N+\tfrac{1}{2})\theta$$

What? Change: $\displaystyle\int_0^{(N+\frac{1}{2})\pi}$

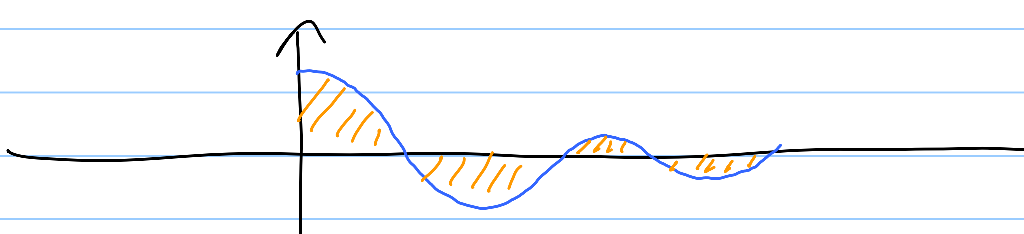
Aha! $= \underbrace{\int_0^{\pi} \left|\frac{\sin\psi}{\psi}\right| d\psi}_{\text{a const. } C} + \int_{\pi}^{N\pi} \left|\frac{\sin\psi}{\psi}\right| d\psi + \underbrace{\int_{N\pi}^{(N+\frac{1}{2})\pi} \left|\frac{\sin\psi}{\psi}\right| d\psi}_{>0}$

$$> C + \int_{\pi}^{N\pi} \left|\frac{\sin\psi}{\psi}\right| d\psi$$

Integral test:

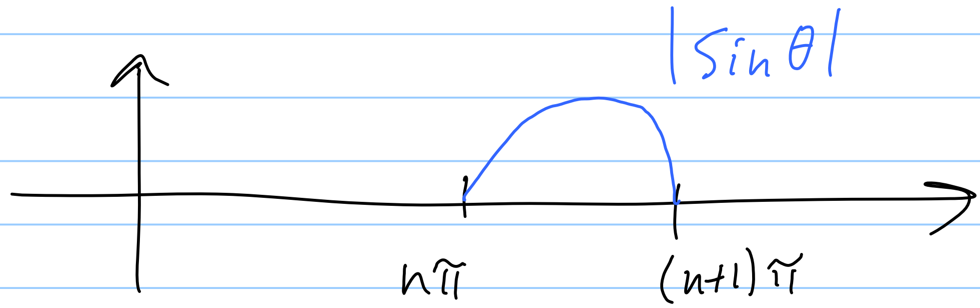


$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} < \underbrace{\int_1^N \frac{1}{t}\, dt}_{\ln N} < \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N-1}$$



$\int_0^{\infty} \frac{\sin\theta}{\theta}\, d\theta$ exists by alternating series test.

But $\int_0^\infty \left|\frac{\sin\theta}{\theta}\right| d\theta = \infty$



$$n\pi < \theta < (n+1)\pi$$

$$\frac{1}{(n+1)\pi} < \frac{1}{\theta} < \frac{1}{n\pi}$$

Mult by $|\sin\theta|$

and $\int_{n\pi}^{(n+1)\pi} d\theta$