Lecture 38 Bessel fcns

Power series sol^{n}s to $y'' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = 1 \cdot a_1 + 2a_2 x + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2 \cdot 1\, a_2 + 3 \cdot 2 a_3 x + \cdots$$

Plug in to the ODE: $y'' + y = 0$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{\overset{\eta}{n-2}}}_{\eta = n-2} + \sum_{n=0}^{\infty} a_n x^n = \overset{\text{want}}{0}$$

$$\sum_{\eta=0}^{\infty} (\eta+2)(\eta+1) a_{\eta+2} x^{\eta}$$

Now change η back to an n

$$\sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1) a_{n+2} + a_n}_{A_n = \frac{f^{(n)}(0)}{n!}} \Big] x^n = \overset{\text{want}}{0}$$

$f \equiv 0$

Recursion formula:

$$\boxed{a_{n+2} = \frac{-a_n}{(n+2)(n+1)}} \qquad n = 0, 1, 2, \ldots$$

<u>Note</u>: $y(0) = a_0$, $y'(0) = a_1$ $\Big\}$ From initial conditions!

<u>$n=0$</u>: $a_2 = \dfrac{-a_0}{2\cdot 1}$

<u>$n=1$</u>: $a_3 = \dfrac{-a_1}{3\cdot 2}$

<u>$n=2$</u>: $a_4 = \dfrac{-a_2}{4\cdot 3} = \dfrac{a_0}{4\cdot 3\cdot 2\cdot 1}$

<u>$n=3$</u>: $a_5 = \dfrac{-a_3}{5\cdot 4} = \dfrac{a_1}{5\cdot 4\cdot 3\cdot 2}$

$\vdots$

Get $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$

$$+ a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \right)$$

$$\boxed{y = a_0 \cos x + a_1 \sin x}$$

Radius of Conv. ?

Ratio Test: $\left| \dfrac{u_{n+1}}{u_n} \right| = \left| \dfrac{u_{\text{far out}}}{u_{\text{right before}}} \right|$

$$= \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \left| \frac{a_{n+2}}{a_n} \right| x^2$$

$$= \left| \frac{-1}{(n+2)(n+1)} \right| x^2 = \frac{x^2}{(n+2)(n+1)}$$

$$\xrightarrow[n\to\infty]{} 0 = L < 1$$

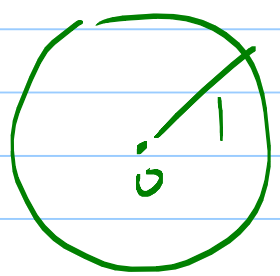
$R = \infty$

So power series converges for all x,

can diff'tiate term by term, see they solve ode!

EX: Get exponential: $y' = y$, $y(0) = 1$

Vibrating circular membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

Take $c=1$.

$$= c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

B.C. $u(1,\theta,t) = 0$

I.C. $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$

Separation of variables: Try

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

Plug into PDE:

$(c=1)$

$$R\Theta T'' = R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T$$

Divide by $R\Theta T$:

$$\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda$$

For today, assume u does not depend on θ. (Radially symm sol^{n}s)

<u>R prob:</u> $u(1,\theta,t) = \underbrace{R(1)}\,\Theta(\theta)T(t) \overset{\text{want}}{=} 0$

Need $\boxed{R(1)=0}$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \lambda$$

<u>Case $\lambda = 0$:</u> $R'' + \frac{1}{r}R' = 0$

Euler eqn. $\longrightarrow$ $r^2R'' + rR' = 0$

$$Ar^2R'' + BrR' + CR = 0$$

Try sol^n r^p :

$$\boxed{Ap(p-1) + Bp + C = 0}$$

For us : $A=1$, $B=1$, $C=0$.

$$p(p-1) + p = 0$$

$$p^2 = 0 \quad \text{Ouch!}$$

$p = 0,0$

Get <u>one</u> sol^n $R_1(r) = r^0 = 1$.

Use Reduction of order to get a second sol^n: Try $R_2(r) = u(r)R_1(r)$.

Plug into ODE. Get a first order simple ODE for u.

Do it : $R_2 = u \cdot 1$

$$r^2 \cdot u'' + r u' = 0$$

$$\frac{dv}{dr} + \frac{1}{r} v = 0 \quad \text{where } v = u'.$$

$$\underbrace{r \frac{dv}{dr} + v}_{\frac{d}{dr}(rv)} = 0$$

So $rv = c$, a const.

$$u' = v = \frac{c}{r}$$

Finally, $u = c \ln r$ and

$$R_2 = u R_1 = c \ln r \cdot 1$$ ← ouch! a drum ripper!

Gen' Sol^n

$$R(r) = c_1 \cdot 1 + c_2 \ln r$$

↑ c_2 must $= 0$.

Need $R(1) = c_1 \cdot 1 \overset{\text{want}}{=} 0$

c_1 must $= 0$ too. Only get zero sol^n.

<u>Case $\lambda > 0$:</u> Can show there are no non-zero bounded sol^{n}s.

<u>Case $\lambda < 0$:</u> Bessel fcns!