MATH 428 Fourier Analysis

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Hwk	100	pts
M.E	100	
FE	200	
	400	pts.

Power series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

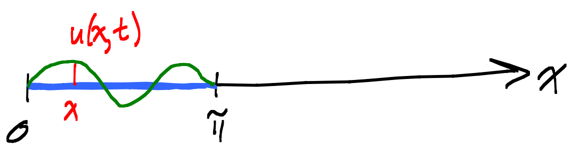
$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$a_n = \frac{f^{(n)}(a)}{n!}$$

Fourier series: $f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx \right)$

on $[-\pi, \pi]$

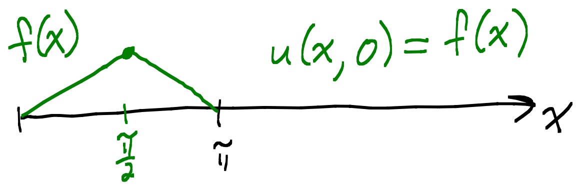
Bernouli vs Bernouli: Vibrating String Prob: Harsichord Prob



Wave eqn:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Time $t=0$



Initial Cond. #1

Hidden Initia Cond #2: Starts from rest

$$\boxed{\frac{\partial}{\partial t} u(x,0) = 0}$$

Jacob's sol^n: $u(x,t) = \varphi(x+ct) + \psi(x-ct)$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct) \right]$$ Wow!

Johann's! Try $u(x,t) = X(x)T(t)$

PDE: Assume $c=1$. Want $\frac{\partial^2}{\partial t^2}[XT] = \frac{\partial^2}{\partial x^2}[XT]$

$$X(x)T''(t) \overset{\text{want}}{=} X''(x)T(t)$$

$$\boxed{\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda}, \quad \text{a constant!}$$

[set $t=\frac{1}{2}$. See LHS = const λ. Now unfix t...]

Get 2 ODE's: $X'' - \lambda X = 0$, $T'' - \lambda T = 0$

Boundary conditions: $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

$u(0,t) = \underline{\underline{X(0)}}T(t) = 0$

$u(\pi,t) = \underline{\underline{X(\pi)}}T(t) = 0$

B.C.'s $\boxed{X(0)=0,\ X(\pi)=0}$

Save I.C.'s for last.

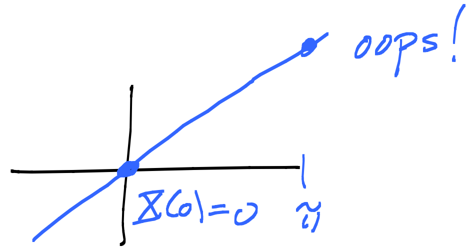
Sturm-Liouville Problem: $X'' - \lambda X = 0$

$$\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

ODE: $X'' - \lambda X = 0$ Try $X(x) = e^{rx}$.

Need $\left(r^2 e^{rx}\right) - \lambda\left(e^{rx}\right) = 0$

$$\underbrace{[r^2-\lambda]}_{\substack{\text{must}\\=0}}\ \underset{\substack{\uparrow\\ \text{never}\\ \text{zero}}}{e^{rx}} = 0$$



Case $\lambda=0$: $X''=0$

So $X'=A$

So $X=Ax+B$

Ouch! Only get $X(x)=0\cdot x+0\equiv 0$ satisfying BC's.

Case $\lambda>0$: $r^2-\lambda=0$ | Let $\lambda=k^2$ $(k>0)$

$r=\pm\sqrt{\lambda}$ | Then $r=\pm k$.

Get 2 sol^{n}s: e^{kx}, e^{-kx}.

Linear ode: So $c_1e^{kx}+c_2e^{-kx}$ is also a sol^n.

Is this the gen^l sol^n? $\leftarrow$ Can I solve any and all initial value probs with these?

[E & U. Thm then $\Rightarrow$ gen^l sol^n ✓]

$$\begin{cases} X(0)=A \\ X'(0)=B \end{cases} \quad \text{want}$$

$X(x)=c_1e^{kx}+c_2e^{-kx}$ | $X(0)=c_1+c_2 \overset{\text{want}}{=} A$

$X'(x)=c_1ke^{kx}-c_2ke^{kx}$ | $X'(0)=c_1k-c_2k=B$

$$\underbrace{\begin{bmatrix}1 & 1\\ k & -k\end{bmatrix}}_{\text{Wronski matrix}}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}A\\B\end{pmatrix}$$

$$\text{Wronskian} = \det\begin{bmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{bmatrix}\Bigg|_{x=0} = -k-k=-2k \leftarrow \text{not zero!}$$

Can get c_1, c_2, no matter

what A, B are.

So $X(x) = c_1 e^{kx} + c_2 e^{-kx}$ is gen' sol^n.

BC's $\quad X(0) = c_1 + c_2 = 0 \leftarrow \boxed{c_2 = -c_1}$

want

$X(\pi) = c_1 e^{k\pi} + c_2 e^{-k\pi} = 0$

$$X(\pi) = c_1 \underbrace{\left[e^{k\pi} - e^{-k\pi} \right]}_{\text{not zero}} = 0$$

$>1 \quad <1$

Ouch! c_1 must $=0$

So $c_2 = 0$ too.

Damn! $X(x) \equiv 0$ again.

Case $\lambda < 0$: $\lambda = -k^2$.

$$r^2 - \lambda = 0$$
$$r^2 - (-k^2) = 0 \qquad r = \pm ki$$
$$r^2 + k^2 = 0$$

Euler: $e^{ikx} = \underbrace{\cos kx}_{u} + i \underbrace{\sin kx}_{v} \leftarrow$ is a complex sol^n to ODE

$$X'' - \lambda X = 0$$

$$[u'' + iv''] - \lambda [u + iv] = 0$$

$$\underbrace{[u'' - \lambda u]}_{=0} + i \underbrace{[v'' - \lambda v]}_{=0} = 0$$

One complex sol^n gives <u>two</u> real sol^{n}s!

linear combos: Get $X(x) = c_1 \cos kx + c_2 \sin kx$ are sol^{n}s.

Wronskian idea shows this is the gen' sol^n!

BC: $X(0) = c_1 \cos 0 + c_2 \sin 0 = \underline{c_1 = 0}$ ← want

$X(\pi) = c_2 \sin k\pi = 0$ ← want