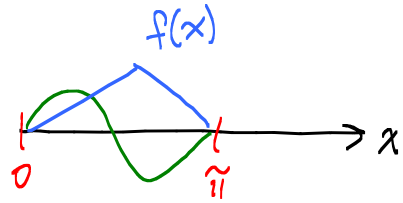
Lecture 2 Harpsichord problem, Johann's sol^n



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases} \quad \text{IC's}$$

$$\text{BC's} \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

Try: $u(x,t) = X(x)T(t)$

$$\boxed{\begin{array}{l} X'' - \lambda X = 0 \\ \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{array}}$$

$$T'' - \lambda T = 0$$

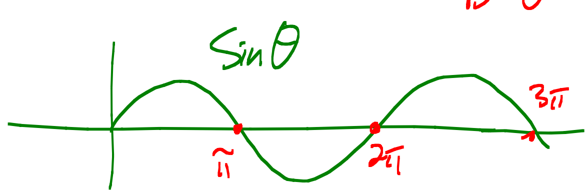
Case $\lambda = 0$: $X(x) \equiv 0$. Nothing but zero sol^n.

Case $\lambda > 0$: Same!

Case $\lambda < 0$: $\lambda = -k^2$. Got $X(x) = \cancel{A\cos kx} + B\sin kx$

$X(0) = A \overset{\text{want}}{=} 0$ $\boxed{A=0}$

$X(\pi) = 0 + \underset{\text{Don't want } B=0}{B}\sin k\pi \overset{\text{want}}{=} 0$ Need $\boxed{\sin k\pi = 0}$



Need $k\pi = n\pi$, $n = 1, 2, 3$

Aha! Get non-zero sol^{n}s when $k = n = 1, 2, 3, \ldots$

For $k=n$: Get $X_n = \sin nx$. Need T_n: $T'' + \underbrace{n^2}_{-k=n^2} T = 0$

Get $T_n(t) = A_n \cos nt + B_n \sin nt$

Get $u_n(x,t) = X_n(x)\, T_n(t)$

PDE ✓
BCs ✓
ICs? No way!

But, since wave is linear, linear combos of sol^n's are sol^n.

Try $u(x,t) = \displaystyle\sum_{n=1}^{\infty} X_n(x)\, T_n(t)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left[A_n \cos nt + B_n \sin nt \right]$$

Need IC's:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \overset{\text{want}}{=} f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin nx \left[-nA_n \sin nt + nB_n \cos nt \right] \Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} nB_n \sin nx \overset{\text{want}}{=} 0$$

Aha! Take all B_n's $= 0$.

Get $\boxed{u(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n \sin nx \cos nt}$

Big problem: Can we find A_n's so that

$$\sum_{n=1}^{\infty} A_n \sin nx = f(x) \, ?$$

Key: $\{\sin nx\}_{n=1}^{\infty}$ is a complete orthogonal set of fcns.

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \frac{\pi}{2} & \text{if } n = m \end{cases}$$

$n=m$: $\displaystyle\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2nx) \, dx = \frac{\pi}{2}$

Idea: $\left[\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_N \vec{u}_N\right] \cdot \vec{u}_m$

$$\vec{v} \cdot \vec{u}_m = c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_m}_{0} + \cdots \underline{\underline{c_m \vec{u}_m \cdot \vec{u}_m}} + \cdots + c_N \underbrace{\vec{u}_N \cdot \vec{u}_m}_{0}$$

Solve for c_m.

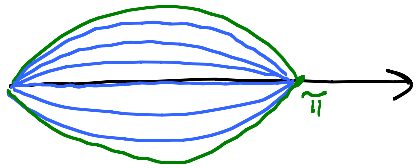
$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\text{all } = 0 \text{ except } n=m \text{ one.}}$$

$$= A_m \int_0^{\pi} \sin^2 mx \, dx = \frac{\pi}{2} A_m$$

Aha! $\boxed{A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$

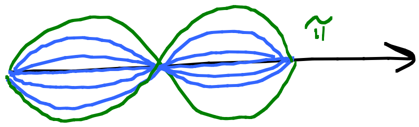
Modes of oscillation: $u_n(x,t) = \sin nx \cos nt$

$n=1$:



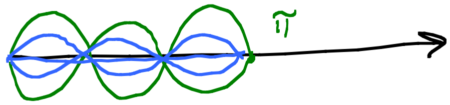
$\sin x \cos t$

$n=2$:

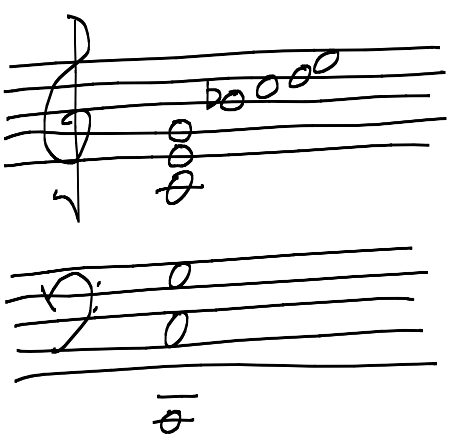


$\sin 2x \cos 2t$

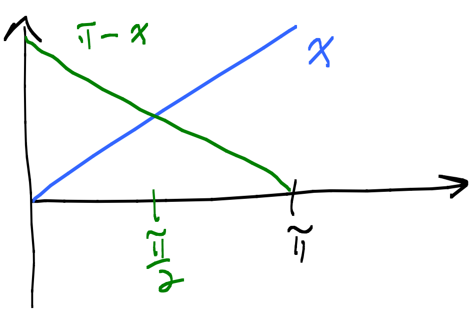
$n=3$:



$\sin 3x \cos 3t$



$n=8$	C
$n=7$	$\sim B^b$
$n=6$	G
$n=5$	E
$n=4$	C
$n=3$	G
$n=2$	C
$n=1$	C



$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$$

$$A_1 = \frac{2}{\pi}\left(\int_0^{\pi/2} x \sin x \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin x \, dx\right)$$

$\vdots$

A_n = Same with nx in place of x.

Get $$u(x,t) = c\left(\frac{1}{1^2}\sin x\cos t - \frac{1}{3^2}\sin 3x\cos 3t + \frac{1}{5^2}\sin 5x\cos 5t - \ldots\right)$$