Lecture 6 Bessel's Inequality

$$f(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-N}^{N} c_n e^{inx}$$

exactly same thing!

$$e^{inx} = \left(e^{ix}\right)^n$$

$$e^{-inx} = \left(e^{-ix}\right)^n$$

Bessel's Ineq: f continuous on $[-\pi, \pi]$, ← real valued

φ_n orthonormal ← real valued

$$\langle \varphi_n, \varphi_m \rangle = \int_{-\pi}^{\pi} \varphi_n \varphi_m \, dx$$ ← no conjugate over φ_m

$$= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Hope: $f \overset{?}{=} \sum_{n=1}^{\infty} c_n \varphi_n$

$$\langle f, \varphi_m \rangle = \sum_{n=1}^{\infty} c_n \langle \varphi_n, \varphi_m \rangle = c_m \cdot 1$$

Required: $\boxed{c_m = \langle f, \varphi_m \rangle}$.

Bessel's idea:

$E =$

$$E^* = \Big\| f - \sum_{n=1}^{N} \overset{A_n}{c_n} \varphi_n \Big\|^2 =$$

$$= \int_{-\pi}^{\pi} \left(f - \sum_{n=1}^{N} \overset{A_n}{c_n} \varphi_n\right)\left(f - \sum_{m=1}^{N} \overset{A_m}{c_m} \varphi_m\right) dx$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^{N} \overset{A_n}{\overset{\downarrow}{c_n}} \underbrace{\int_{-\pi}^{\pi} f\,\varphi_n\,dx}_{c_n} - \sum_{m=1}^{N} \overset{A_m}{\overset{\downarrow}{c_m}} \underbrace{\int_{-\pi}^{\pi} f\,\varphi_m\,dx}_{c_m}$$

$$+ \sum_{n=1}^{N} \sum_{m=1}^{N} \overset{A_n}{\overset{\downarrow}{c_n}}\,\overset{A_m}{\overset{\downarrow}{c_m}} \underbrace{\int_{-\pi}^{\pi} \varphi_n \varphi_m\,dx}_{\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}}$$

$$= \int_{-\pi}^{\pi} f^2\,dx - S - S + S \quad \text{where} \quad S = \sum_{1}^{N} c_n^2$$

Hmmm.

$$E^* = \underbrace{\left\| f - \sum_{1}^{N} c_n \varphi_n \right\|^2}_{\geq 0} = \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^{\infty} c_n^2$$

Aha!

$$\sum_{n=1}^{N} c_n^2 \leq \int_{-\pi}^{\pi} f^2\,dx$$

Aha! Can let $N \to \infty$! Get

$$\boxed{\sum_{n=1}^{\infty} c_n^2 \leq \int_{-\pi}^{\pi} f^2\,dx}$$

<u>Notice</u>:

$$|b_n| = \left| \int_{-\pi}^{\pi} \underbrace{f(x)}_{|f(x)| \leq M} \underbrace{\sin nx}_{|\sin nx| \leq 1}\,dx \right| \leq M \cdot 1 \cdot 2\pi$$

Cor: Fourier coeff are square summable and they $\to 0$ as $n \to \infty$!

Fact: "Generalized Fourier" expansion $\sum_{n=1}^{N} c_n \varphi_n$ is the best approx to f in L^2-norm among fcns of form $\sum_{n=1}^{N} A_n \varphi_n$.

$$E - E^* = \text{write it out} = \underbrace{\sum_{n=1}^{N} (A_n - c_n)^2}_{\text{sum of squares}} \geq 0$$

Only equal if $A_n = c_n$ for $n = 1, \dots, N$.

Def^n: If $\sum_{n=1}^{\infty} c_n^2 = \int_{-\pi}^{\pi} f^2 \, dx$ for continuous f's, then $\{\varphi_n\}_{n=1}^{\infty}$ is called a complete orthonormal basis for $L^2[-\pi, \pi]$. [Bessel's ineq $\to$ Parseval identity.]

Facts: $\{\sin nx\}_{n=1}^{\infty}$ is complete orthog basis for $L^2[0, \pi]$.

So is $\{1, \cos nx : n = 1, 2, \dots, \infty\}$.

$\{1, \cos nx, \sin nx : n = 1, 2, \dots, \infty\}$ is complete $L^2[-\pi, \pi]$.

Hilbert space: $L^2[-\pi,\pi]$ = space of square integrable fcns on $[-\pi,\pi]$.

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f\, g\, dx \qquad \|f\| = \sqrt{\langle f,f\rangle}$$

$$\|f-g\|^2 = \int_{-\pi}^{\pi} (f-g)^2\, dx$$

Remark: φ_n are always linearly independent:

$$\underbrace{\langle \varphi_m, \sum c_n \varphi_n\rangle}_{c_m = 0 \ \checkmark} = \langle \varphi_m, 0\rangle = 0$$

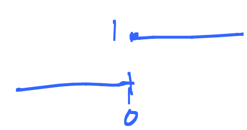
Think: $f = \sum_{n=1}^{\infty} c_n \varphi_n$

c_n ← coord of f in "φ_n direction"

Fact: $L^2[-\pi,\pi] \approx \ell^2$ ← square summable series $\{c_n\}_{n=1}^{\infty}$

Isomorphism: $\int_{-\pi}^{\pi} f^2\, dx = \sum_{1}^{\infty} c_n^2$ $\qquad \langle c_n, b_n\rangle = \sum_{1}^{\infty} c_n b_n$

Maple: > u := x -> Heaviside(x); ↵



> f := x ->

[u(x-Pi/4) - u(x-Pi/2)]*(x-Pi/4) + ...

↑ on at $\frac{\pi}{4}$ ↑ off at $\frac{\pi}{2}$

> F := x -> f(x) - f(-x)