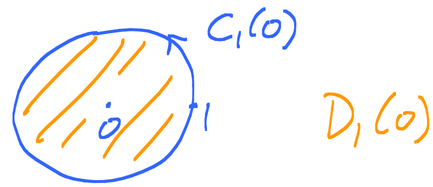
Lecture 8 Dirichlet problem on unit disc

Given f on $C_1(0)$. Find

u continuous on $\overline{D_1(0)}$,

u harmonic in $D_1(0)$,

$u=f$ on $C_1(0)$



$$\Theta'' + \lambda \Theta = 0 \quad (*)$$

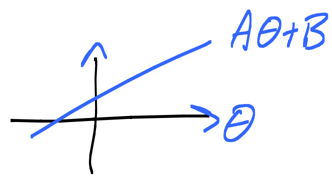
$$\begin{cases} \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta'(\pi) \end{cases} \leftarrow \text{Periodic boundary conditions} \leftarrow$$

allows to extend $\Theta(\theta)$ to $\mathbb{R}$ as a periodic sol^n to $(*)$.

Case $\lambda = 0$: $\Theta'' \equiv 0$

$\Theta' = A$

$\Theta(\theta) = A\theta + B$



Aha! Need $A=0$.

$\Theta(\theta) = B$, a const, is periodic.

R problem: $(\lambda = 0)$
$$r^2 R'' + rR' - \underbrace{\lambda}_{=0} R = 0$$

$$R'' + \frac{1}{r} R' = 0 \leftarrow \text{Let } v = R'(r).$$

$$\frac{dv}{dr} + \frac{1}{r} v = 0 \leftarrow \text{Linear first order homog ODE.}$$

Integrating factor: $\mu = e^{\int \frac{1}{r} dr} = e^{\ln r} = r$

Multiply by $\mu = r$: $\underbrace{r\frac{dv}{dr} + v}_{\frac{d}{dr}[\mu v]} = 0$

So $rv \equiv c$, a const.

$$R' \to v = \frac{c}{r}$$

$$R(r) = \int \frac{c}{r}\,dr = \underbrace{C \ln r}_{} + B_2$$

Oy! blows "up" at $r=0$.

Physically meaningful: Need $C=0$.

Found a sol^n: $u(r,\theta) = \Theta(\theta)R(r) = B \cdot B_2$, const.

Take $u_0(r,\theta) = 1$.

<u>Case $\lambda < 0$</u>: Write $\lambda = -k^2$. $\Theta'' - k^2\Theta = 0$.

Try $\Theta(\theta) = e^{r\theta}$: $r^2 e^{r\theta} - k^2 e^{r\theta} = 0$ ← want

$$(r^2 - k^2)\, e^{r\theta} = 0$$

never zero

Need $r = \pm k$.

Get $\Theta(\theta) = c_1 e^{k\theta} + c_2 e^{-k\theta}$ or $= A_1 \underbrace{\cosh k\theta}_{\frac{e^{k\theta}+e^{-k\theta}}{2}} + A_2 \underbrace{\sinh k\theta}_{\frac{e^{k\theta}-e^{-k\theta}}{2}}$

Intuition: No way can we make these periodic.

Work: Set
$$\left.\begin{aligned} \Theta(-\pi) &= \Theta(\pi) \\ \Theta'(-\pi) &= \Theta'(\pi) \end{aligned}\right\} \leftarrow \text{2 linear equs in } c_1, c_2.$$

Sol^n: $c_1, c_2 = 0$.

Only get zero sol^n. $u \equiv 0$.

<u>Case $\lambda > 0$</u>: $\Theta'' + k^2 \Theta = 0$

$\lambda = k^2$

Gen'l Sol'n: $\Theta(\theta) = c_1 \cos k\theta + c_2 \sin k\theta$

Set periodic conditions: Conclude that you only get periodic sol'ns when $\cos k\theta$ and $\sin k\theta$ are 2π-periodic, and this only happens when $k = 1, 2, 3, \ldots$

For $n = 1, 2, 3, \ldots$, Get $\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$.

R problem for $\lambda = n^2$: $\underbrace{r^2 R'' + r R' - n^2 R = 0}_{\text{Euler equation}}$

Euler: Try $R(r) = r^p$. Plug in:

want

$$r^2 \cdot (\underbrace{p(p-1) r^{p-2}}_{R''}) + r \cdot (\underbrace{p r^{p-1}}_{R'}) - n^2 \underbrace{r^p}_{R} = 0$$

$$\Big[\underbrace{p(p-1) + p - n^2}_{\text{need } = 0}\Big] r^p = 0$$

$$p^2 - n^2 = 0$$

$$\boxed{p = \pm n}$$

Get $R(r) = c_1 r^n + \underbrace{c_2 r^{-n}}_{\text{Ouch!}}$ Must have $c_2 = 0$ to be physically meaningful

Got $R_n(r) = r^n$

We found $u_n(r,\theta) = r^n\left(a_n \cos n\theta + b_n \sin n\theta\right)$

$\Delta u \equiv 0 \leftarrow$ Linear homogeneous PDE. $\quad \Delta[c_1u_1 + c_2u_2] = c_1\Delta u_1 + c_2\Delta u_2$

Get $\quad u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n\left(a_n \cos n\theta + b_n \sin n\theta\right)$

Last part. Worry about boundary condition

$$u(1,\theta) = \underbrace{a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)}_{\text{"Full" Fourier series.}} \overset{\text{want}}{=} f(\theta)$$

<u>Sol^n</u>:

$$a_0 = \text{Ave of } f = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$$

<u>HWk 2: Problem 1.</u> Think $\quad f(x) \overset{?}{=} \sum_{n=1}^{\infty} A_n \sin nx$

Related to

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$f'(x) \overset{?}{=} \sum_{n=1}^{\infty} \underbrace{n A_n}_{B_n} \cos nx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f'(x)\cos nx\,dx = \text{Int. by parts} = n A_n$$

Bessel's Ineq. $f = \sum_1^\infty c_n \varphi_n$ ← orthonormal

$$\sum_{n=1}^\infty |c_n|^2 \le \int_{-\pi}^{\pi} |f|^2\,dx$$

$$\frac{\cos nx}{\sqrt{\frac{\pi}{2}}} \leftarrow \text{orthonormal} \rightarrow \frac{\sin nx}{\sqrt{\frac{\pi}{2}}}$$

Cosine series: $$\sum_{n=1}^\infty (nA_n)^2 \le \int |f'|^2$$

$$\sum_{n=1}^\infty A_n^2 \le \int |f|^2$$

Wow! $\sum_1^\infty (nA_n)^2$ finite $\Rightarrow$ $A_n \to 0$ really fast.

Conclude that $\sum_1^\infty A_n \sin nx \xrightarrow[\text{uniformly}]{}$