Lecture 12 Féjér's Theorem

HWK 4 on home page
due Friday next week

Thm: Suppose f is continous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$.

Given $\varepsilon > 0$, there is a trigonometric polynomial

$$T_N(x) = \sum_{n=-N}^{N} \tilde{c}_n e^{inx} \quad \text{such that}$$

$$|f(x) - T_N(x)| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Strategy: Solve D. Prob with boundary data $f(\theta)$.

Get $u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|re^{i\theta} - e^{it}|^2} f(t)\,dt.$

Know $u(r,\theta) \to f(\theta)$ as $r \nearrow 1$ uniformly on $[-\pi, \pi]$.

Let $T_N(\theta) = \sum_{-N}^{N} c_n r^{|n|} e^{in\theta}$ where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt$

Let $\varepsilon > 0$. Get an r close to one such

$$|u(r,\theta) - f(\theta)| < \frac{\varepsilon}{2}.$$

Claim: Can choose N so that $|u(r,\theta) - T_N(\theta)| < \frac{\varepsilon}{2}$.

Why: 1) $|c_n| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(t) e^{-int}\,dt\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)| \cdot \underbrace{|e^{-int}|}_{=1}\,dt$

$$\leq \frac{1}{2\pi}\underbrace{\left(\operatorname*{Max}_{[-\pi,\pi]}|f|\right)}_{M}\cdot 2\pi \leq M$$

2) $$\left|u(r,\theta)-T_N(\theta)\right| = \left|\sum_{|n|>N} c_n r^{|n|} e^{in\theta}\right|$$

$$\leq \sum_{|n|>N} \underbrace{|c_n|}_{\leq M} r^{|n|} \underbrace{\left|e^{in\theta}\right|}_{1}$$

$$\leq \underbrace{M\left(\sum_{n=N+1}^{\infty} r^n + \sum_{n=N+1}^{\infty} r^n\right)}_{2M r^{N+1} \underbrace{\sum_{n=0}^{\infty} r^n}_{\frac{1}{1-r}}} \leq \frac{2Mr^{N+1}}{1-r}$$

$\longrightarrow 0$ as $N \to \infty$ because $0<r<1$

Done! $$|f(\theta)-T_N(\theta)| = |f(\theta)-u(r,\theta)+u(r,\theta)-T_N(\theta)|$$

$$\leq \underbrace{|f-u|}_{<\frac{\varepsilon}{2}} + \underbrace{|u-T_N|}_{<\frac{\varepsilon}{2}}$$ if N big enough.

Big consequence: Féjér's $\Longrightarrow$ Parseval's equality.

Review Bessel's Ineq: $f = \sum_{n=1}^{\infty} c_n \varphi_n$ $\qquad \langle \varphi_n, \varphi_m \rangle = \delta_{nm}$

$= \int_\alpha^\beta \varphi_n(x)\overline{\varphi_m(x)}\,dx$

Fourier coeff $\rightarrow$ $c_n = \langle f, \varphi_n \rangle$. $\qquad \|f\|^2 = \int_\alpha^\beta |f|^2\,dx$

$$\|f\|^2 - \sum_{n=1}^N |c_n|^2 = \underbrace{\int_\alpha^\beta \Big|f - \sum_{n=1}^N c_n \varphi_n\Big|^2 dx}_{\geq 0} \leq \int_\alpha^\beta \Big|f - \sum_{n=1}^N A_n \varphi_n\Big|^2 dx$$

non Fourier coeff

$\Rightarrow$ Bessel's: $\displaystyle\sum_{n=1}^N |c_n|^2 \leq \|f\|^2$

Let $N \to \infty$. See

$$\sum_{n=1}^\infty |c_n|^2 \leq \|f\|^2$$

shows Fourier coeff $\rightarrow 0$ as $|n| \to \infty$.

Big wow! Use T_N on this side. Make RHS as small as desired!

Conclude

$$0 \leq \|f\|^2 - \sum^N |c_n|^2 \leq \varepsilon$$

need big enough N here. $\qquad$ arb.

Conclude

Parseval's equality: $\displaystyle\sum_{n=-\infty}^\infty |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\,d\theta$

Fourier coeff

Remark: $\displaystyle f \sim a_0 + \sum_{n=1}^\infty (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-\infty}^\infty c_n e^{in\theta}$

$c_n = \dfrac{a_n + i b_n}{2}$, $n > 0$ $\qquad a_0 = c_0$.

$c_n = \dfrac{a_{|n|} - i b_{|n|}}{2}$, $n < 0$

So $$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta$$

$$\boxed{2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta}$$

<u>Remark:</u> Fourier Series on $[-L, L]$.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

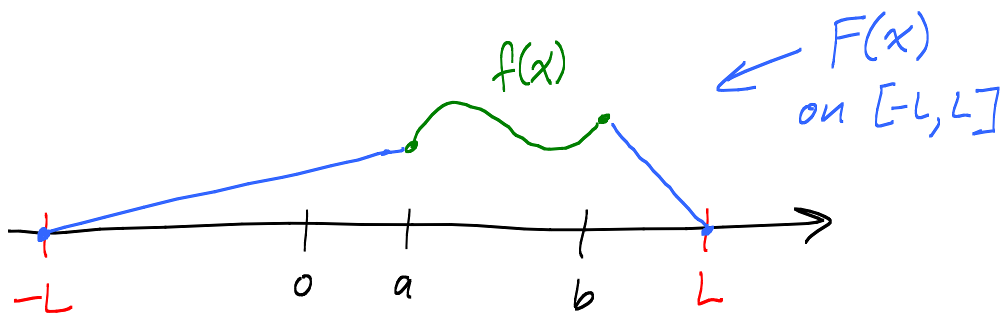
$$= \sum_{-\infty}^{\infty} c_n e^{i n\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-in\pi x/L}\,dx$$

Same theory!

<u>Féjér's $\Rightarrow$ Weierstraß approximation theorem:</u>

Suppose f is continuous on $[a,b]$. Given $\varepsilon > 0$, there a polynomial $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ on $[a,b]$.

Pf:



Fejer's: Get trig poly T_N with $|T_N - F| < \frac{\varepsilon}{2}$ on $[-L, L]$. Idea

$$e^{in\pi x/L} = \cos + i \sin$$

$$e^{-in\pi x/L} = \cos - i \sin$$

approx by truncated power series on $[-L, L]$.