Lecture 13 The big day! $S_N \to f$

Thm: Suppose f is continuous and $f'(x)$ exists. Then $S_N(x) \to f(x)$.

Assume f is 2π-periodic and continuous.

$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx}$$

where $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

$$S_N(x) = \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} \cdot e^{inx}\, dt \right)$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-t)} \right)}_{D_N(x-t)} f(t)\, dt$$

$D_N(x-t) \leftarrow$ Dirichlet kernel

Dirichlet kernel: $$D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx}$$

Properties:

1) $$D_N(x) = \frac{1}{2\pi} \frac{\sin (N+\frac{1}{2})x}{\sin \frac{x}{2}}$$

2) L'H $\Rightarrow$ $$D_N(0) = \frac{1}{2\pi} \frac{N+\frac{1}{2}}{\frac{1}{2}} = \frac{N+\frac{1}{2}}{\pi}$$

3) $$\int_{-\pi}^{\pi} D_N(x)\, dx = 1$$

4) D_N is $\underbrace{\text{even}}_{\uparrow \text{ from (1)}}$ and 2π-periodic

Pf: (3): $\displaystyle\int_{-\pi}^{\pi} D_N\,dx = \int_{-\pi}^{\pi} \frac{1}{2\pi}\sum_{-N}^{N} e^{inx}\,dx$

$$= 0+0+\cdots+0+\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx + 0 + \cdots + 0 \quad \checkmark$$

Famous calculation:

$$2\pi\, D_N(x) = \underbrace{\cdots +}_{e^{-ix}+e^{-i2x}+\cdots+e^{-iNx}} 1 + e^{ix} + e^{i2x} + \cdots + e^{iNx}$$

$$= \underbrace{\left[(e^{-ix}) + (e^{-ix})^2 + \cdots + (e^{-ix})^N\right]}_{e^{-ix}\left[1 + (e^{-ix}) + \cdots + (e^{-ix})^{N-1}\right]} + \underbrace{\Big[1 + \underbrace{(e^{ix})}_{r} + \cdots + (e^{ix})^N\Big]}_{\frac{1-r^{N+1}}{1-r}}$$

$$= \underbrace{e^{-ix}}_{=\frac{1}{e^{ix}}} \frac{1-(e^{-ix})^N}{1-e^{-ix}} + \frac{1-(e^{ix})^{N+1}}{1-e^{ix}}$$

$$= \frac{1-e^{-iNx}}{e^{ix}-1} + \frac{1-e^{i(N+1)x}}{1-e^{ix}}$$

$$= \frac{e^{-iNx}-e^{i(N+1)x}}{1-e^{ix}} \cdot \frac{e^{-ix/2}}{e^{-ix/2}} \cdot \frac{(-1)}{(-1)}$$

$$= \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} \cdot \frac{\left(\frac{1}{2i}\right)}{\left(\frac{1}{2i}\right)}$$

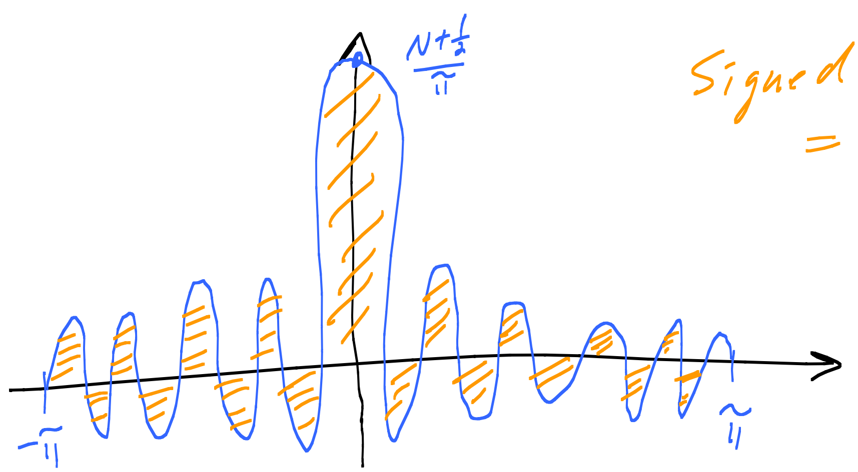
$$= \frac{\sin (N+\frac{1}{2})x}{\sin \frac{x}{2}} \checkmark$$

Remark: $\underbrace{\sum_{-N}^{-1} e^{inx}}_{\sum_{n=1}^{N} \underbrace{e^{-inx}}_{\cos nx - i \sin nx}} + \sum_{0}^{N} \underbrace{e^{inx}}_{\cos nx + i \sin nx}$

So $2\pi\, D_N(x) = 1 + 2 \sum_{n=1}^{N} \cos nx$

Famous graph:

Gigantic near 0.



Signed area $= 1$

Bad things: $\int_{-\pi}^{\pi} |D_N(x)|\, dx \longrightarrow \infty$ as $N \to \infty$.

$D_N(x)$ not a positive kernel.

Good thing: $D_N(x)$ wiggles like hell.

Pf: $\underbrace{S_N(x)}_{} - f(x)$

$$\int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tau}) f(t)\,dt - f(x)\underbrace{\int_{-\pi}^{\pi} D_N(t)\,dt}_{=1}$$

$$\begin{cases} \tau = x - t \Leftarrow t = x - \tau \\ d\tau = -dt \end{cases}$$

When $t = -\pi$; $\tau = x + \pi$

$t = \pi$; $\tau = x - \pi$

Get $\displaystyle\int_{\tau = x+\pi}^{x-\pi} D_N(\tau) f(x-\tau)\,(-d\tau)$

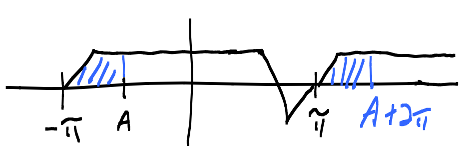
$$= \int_{x-\pi}^{x+\pi} D_N(\tau) f(x-\tau)\,d\tau$$

change τ to t.

combine with

$$-\int_{-\pi}^{\pi} D_N(t) f(x)\,dt$$

Key fact: 2π periodic fcns: $\displaystyle\int_{-\pi}^{\pi} h\,dt = \int_{\substack{\text{Interval}\\ \text{of length}\\ 2\pi}} h\,dt$



So:

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t)\left[f(x-t) - f(x)\right] dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(N+\tfrac{1}{2})t \underbrace{\left[\frac{f(x-t) - f(x)}{\sin\frac{t}{2}}\right]}_{g(t).} dt$$

$g(0) = $ L'H

$$\sin(N+\tfrac{1}{2})t = \sin Nt \cos\tfrac{t}{2} + \cos Nt \sin\tfrac{t}{2}$$

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin Nt \left[\underbrace{g(t)\cos\tfrac{t}{2}}\right] dt \leftarrow \text{ b_N coeff for}$$

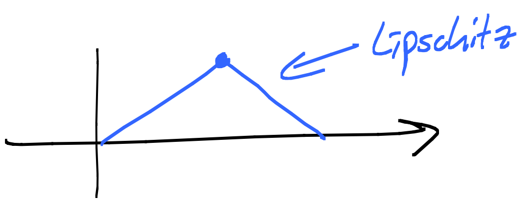
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos Nt \left[\underbrace{g(t)\sin\tfrac{t}{2}}\right] dt \leftarrow \text{ a_N coeff for}$$

Bessel's Ineq showed a_N and $b_N \to 0$ as $N \to \infty$!

So $S_N(x) \to f(x)$ where f diff'ble (plus contin.).

Remark: Can replace diff'ble by Lipschitz:

$$|f(x) - f(t)| \le C|x - t|$$



Hmmm. What is rate of conv?

$c_n \sim f$

$inc_n \sim f'$

$-n^2c_n \sim f''$

Key: f' continuous:

Bessel's $\sum |nc_n|^2 < \infty$

f'' cont.: $\sum |n^2 c_n|^2 < \infty$