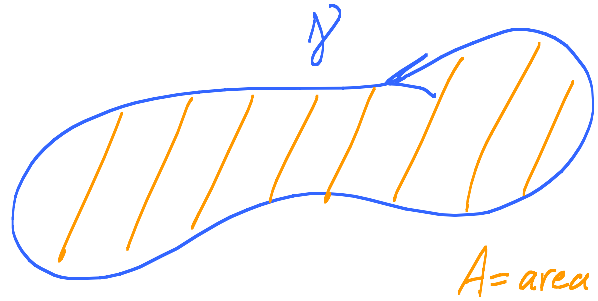
Lecture 22 Applications of Fourier series

HWK 6 due Friday after break

The Isoparametric Inequality



$$A \le \frac{L^2}{4\pi}$$

A = area

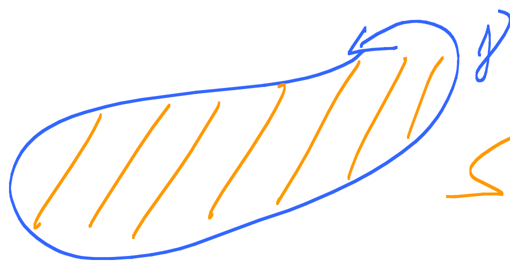
γ length L

Equality only if γ = circle.

Circle case: $A = \pi r^2 \quad L = 2\pi r \leftarrow r = \frac{L}{2\pi}$

$$A = \pi \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

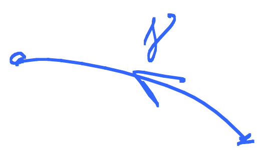
Review Green's Thm



Ω

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

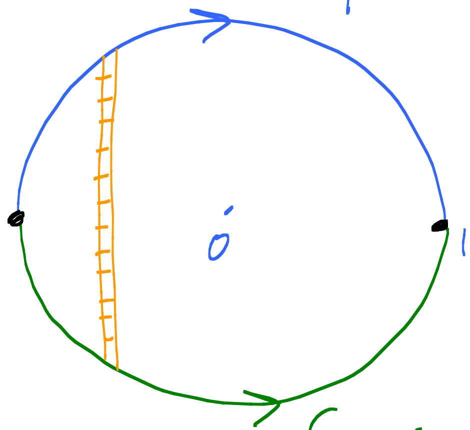
Pf: In case Ω = disc.



γ: $x = x(t)$, $y = y(t)$ $\quad a \le t \le b$

$$\int_\gamma P\,dx = \int_a^b \underbrace{P(x(t), y(t))}_{P(x,y)} \underbrace{\dot{x}(t)\,dt}_{dx = \frac{dx}{dt}\cdot dt}$$

C_{top} : $x(t) = t$, $y_{top}(t) = \sqrt{1-t^2}$ $\quad -1 \le t \le 1$



C : Counterclockwise around circle $= C_{bot}$ followed by $-C_{top}$

C_{bot} : $x(t) = t$, $y_{bot}(t) = -\sqrt{1-t^2}$ $\quad -1 \le t \le 1$

$$\int_C P\,dx = \left(\int_{C_{bot}} + \int_{-C_{top}} \right) P\,dx = \int_{C_{bot}} P\,dx - \int_{C_{top}} P\,dx$$

$$= \int_{-1}^{1} \Big(P(t, y_{bot}(t)) - P(t, y_{top}(t)) \Big) \underbrace{1}_{\dot{x}(t)} \cdot\, dt$$

$$= -\int_{-1}^{1} \underbrace{P(x, y_{top}(x)) - P(x, y_{bot}(x))}_{\int_{y_{bot}(x)}^{y_{top}(x)} \frac{\partial P}{\partial y}(x,y)\,dy} \, dx \quad \leftarrow \text{change } t \text{ to } x.$$

$$= -\int_{x=-1}^{1} \left(\int_{y=y_{bot}(x)}^{y_{top}(x)} \frac{\partial P}{\partial y} \, dy \right) dx$$

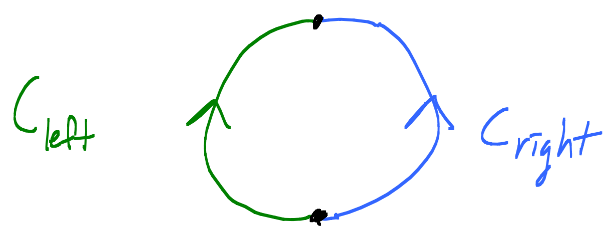
$$= -\iint_{D_1(0)} \frac{\partial P}{\partial y} \, dx\, dy$$

Other half: $$\int_C Q\,dy = \iint_{D_1(0)} \frac{\partial Q}{\partial x}\,dx\,dy$$

Proof same <u>except</u> put "left" and "right" in place of "top" and "bottom".

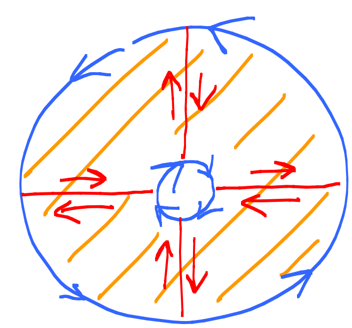
$y(t) = t \quad -1 \le t \le 1$

$x_{right}(t) = \sqrt{1-t^2}$



<u>Remark</u>: Works for any γ that has a top, bottom, left, right, <u>and</u> any γ that you can cut up to have pieces with t, b, l, r property.

<u>EX</u>:



$$\int_{boundary} P\,dx + Q\,dy = \iint_{▨} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Red curves cancel when add up 4 quads!

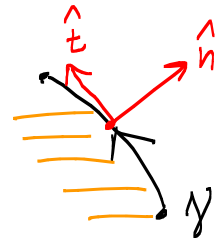
<u>Other versions of Green's</u>:

Divergence Theorem: $$\int_\gamma \vec{F} \cdot \hat{n}\,ds = \iint_\Omega \mathrm{Div}\,\vec{F}\,dx\,dy$$

Green's Identities: $$\int_\gamma u \frac{\partial v}{\partial n}\,ds = \iint_\Omega (\nabla u \cdot \nabla v + u \Delta v)\,dx\,dy$$

$$\int_\gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \iint_\Omega (u\Delta v - v\Delta u)\, dx\,dy$$

Divergence Thm:



γ: $x(t), y(t)$

t = arc length.

$\hat{t} = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}$

unit tangent vector

$\hat{n} = \dot{y}(t)\hat{\imath} - \dot{x}(t)\hat{\jmath}$

unit normal vector

Recall vector $\perp$ (a,b) is $(-b, a)$ or $(b,-a)$.

$\vec{F} = P\hat{\imath} + Q\hat{\jmath}$

$$\int_\gamma \vec{F}\cdot\hat{n}\, ds = \int_a^b \left[P(x(t),y(t))\hat{\imath} + Q(x(t),y(t))\hat{\jmath}\right]\cdot\left[\dot{y}(t)\hat{\imath} - \dot{x}(t)\hat{\jmath}\right] dt$$

$$= \int_\gamma P\,dy - Q\,dx$$

$$= \iint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \iint_\Omega \operatorname{div}\vec{F}\, dx\,dy \checkmark$$

Green's #1:

$$\int_\gamma u\frac{\partial v}{\partial n}\, ds = \int_\gamma \underbrace{u\nabla v}_{\vec{F}}\cdot\hat{n}\, ds \underset{\text{Div Thm}}{=} \iint_\Omega \underbrace{\operatorname{div}(u\nabla v)}_{}\, dx\,dy$$

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)$$

$$\nabla u\cdot\nabla v + u\Delta v \checkmark$$

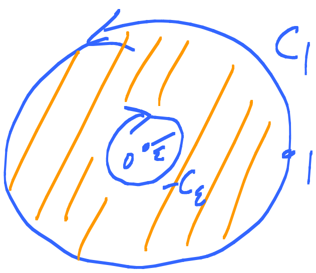
Green's #2:

$$\int_\gamma u \frac{\partial v}{\partial n}\, ds = \iint_\Omega (\nabla u \cdot \nabla v + u \Delta v)\, dx\,dy$$

$$\int_\gamma v \frac{\partial u}{\partial n}\, ds = \iint_\Omega (\nabla v \cdot \nabla u + v \Delta u)\, dx\,dy$$

subtract $\qquad$ $\nabla u \cdot \nabla v$ parts cancel. ✓

Fact: Harmonic fcns satisfy the averaging property.



$u =$ given harmonic fcn.

$v = \ln r = \ln\sqrt{x^2+y^2}$ $\leftarrow$ harmonic

Green's #2:

$$\int_{C_1} u \underbrace{\frac{\partial v}{\partial n}}_{=1}\, ds + \int_{C_1} v \frac{\partial u}{\partial n}\, ds$$

$v = 0$ on C_1

$$\underbrace{-\int_{C_\varepsilon} u \underbrace{\frac{\partial v}{\partial n}}_{1/\varepsilon}\, ds}_{\to\, 2\pi u(0) \text{ as } \varepsilon \to 0} \underbrace{- \int_{C_\varepsilon} v \frac{\partial u}{\partial n}\, ds}_{\to\, 0 \text{ as } \varepsilon \to 0}$$

v: $\ln \varepsilon$

curve length $2\pi\varepsilon$

$$\iint_{\text{Annulus}} u\Delta v - v\Delta u \; dx\,dy = 0 \qquad \text{both harmonic.}$$

Get $\quad u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\cos\theta, \sin\theta)\, d\theta$