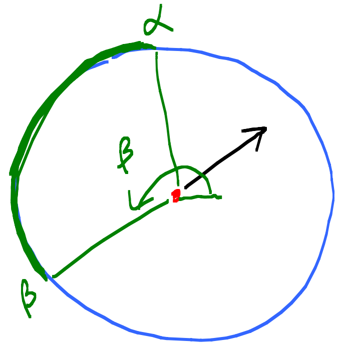
Lecture 25 Weyl's Equidistribution Theorem

HWK 6 due Fri.



Spin needle randomly

$$\frac{\#\text{hits in first quad}}{\#\text{spins}} \sim \frac{1}{4}$$

Herman Weyl (Vile)

André Weil (Vay)

$$\frac{\#\text{hits in } (\alpha,\beta)}{\#\text{spins}} \sim \frac{\beta-\alpha}{2\pi}$$

Thm: If γ is an irrational multiple of π, then $e^{in\gamma}$ are equidistributed on the unit circle, i.e.,

$$R_N = \frac{\#\text{times } e^{in\gamma} \text{ falls in } (\alpha,\beta),\ n=1,2,3,\dots,N}{N}$$

$$\longrightarrow \frac{\beta-\alpha}{2\pi} \quad \text{as} \quad N \to \infty.$$

Pf: $$\chi = \chi_{(\alpha,\beta)}(\theta) = \begin{cases} 1 & e^{i\theta} \in \{e^{it} : \alpha \le t \le \beta\} \\ 0 & \text{otherwise} \end{cases}$$

2π periodic

$$= \begin{cases} 1 & \theta \in (\alpha,\beta) \text{ modulo } 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$R_N = \frac{1}{N} \sum_{n=1}^{N} \chi(e^{in\gamma})$$

Hmmm. Looks like a Riemann sum!

$$= \frac{1}{2\pi} \cdot \frac{2\pi}{N} \sum_{n=1}^{N} \xrightarrow{?} \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi \, d\theta = \frac{\beta-\alpha}{2\pi}$$

$\uparrow$
$\Delta\theta$

Lemma: If f is continuous and 2π-periodic, then

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

Pf: **Step 1:** True for Fourier series basis fcns $e^{im\theta}$.

Check for $f(\theta) \equiv 1$:

$$\frac{1}{N}\sum_{n=1}^{N} 1 = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,d\theta \quad \checkmark$$

Check for $f(\theta) = e^{im\theta}$, $m \in \mathbb{Z} - \{0\}$.

$$\underbrace{\frac{1}{N}\sum_{n=1}^{N} e^{inm\gamma}} \xrightarrow{?} \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{im\theta}}_{\perp 1}\,d\theta = 0$$

$$\frac{1}{N}\cdot e^{im\gamma} + (e^{im\gamma})^2 + \cdots + (e^{im\gamma})^N$$

$$\frac{1}{N}\cdot e^{im\gamma}\cdot \underbrace{\left[1 + (e^{im\gamma}) + \cdots + (e^{im\gamma})^{N-1}\right]} = \frac{1}{N}\,\frac{\overbrace{e^{im\gamma} - e^{im(N+1)\gamma}}^{|\cdot|\text{ bounded by } 2}}{1 - e^{im\gamma}}$$

$$\frac{1}{N}\cdot e^{im\gamma}\cdot \frac{1-(e^{im\gamma})^{N-1+1}}{1-e^{im\gamma}}$$

$\frac{1}{N}$ makes it $\to 0$ as $N \to \infty$

Key: $e^{im\gamma} \neq 1$ for any m!

Step 2: Next, see true for Trig Polys!

Step 3: Given continuous 2π-periodic fcn f (let $\varepsilon > 0$),

$\exists$ Trig Poly $P(\theta)$ with $|f(\theta) - P(\theta)| < \frac{\varepsilon}{3}$

for all θ. (Fejér's Thm)

$$\mathcal{E}_N = \left| \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \right|$$

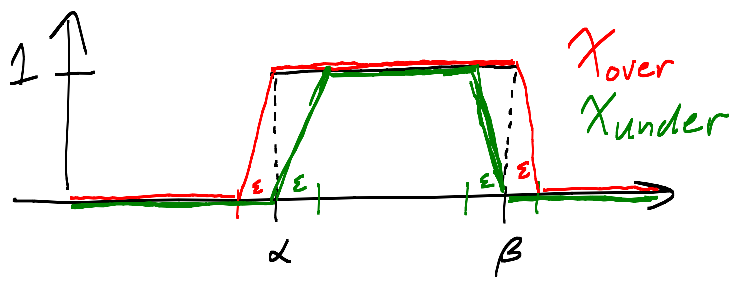
$$= \left| \underbrace{\frac{1}{N}\sum_{n=1}^{N}\big(f(n\gamma) - P(n\gamma)\big)}_{T_1} + \frac{1}{N}\sum_{n=1}^{N} P(n\gamma) + \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)\, d\theta - \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(P(\theta) - f(\theta)\big)\, d\theta}_{T_3} \right|$$

$$|T_1| \le \underbrace{\frac{1}{N}\sum_{n=1}^{N} \frac{\varepsilon}{3}}_{=\frac{\varepsilon}{3}} \qquad |T_3| \le \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varepsilon}{3}\, d\theta}_{=\frac{\varepsilon}{3}}$$

Have $$|\mathcal{E}_N| \le \frac{2\varepsilon}{3} + \underbrace{\left| \frac{1}{N}\sum_{n=1}^{N} P(n\gamma) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\theta)\, d\theta \right|}_{\to 0 \text{ as } N \to \infty, \text{ by Lemma.}}$$

Pick N_0 so this last term is $< \frac{\varepsilon}{3}$ when $N > N_0$.

Last step:



$$\chi_u \leq \chi \leq \chi_o$$

$$\frac{1}{N}\sum_{N=1}^{N} \chi_u(e^{in\gamma}) \leq \frac{1}{N}\sum_{n=1}^{N} \chi(e^{in\gamma}) \leq \frac{1}{N}\sum_{n=1}^{N} \chi_o(e^{in\gamma})$$

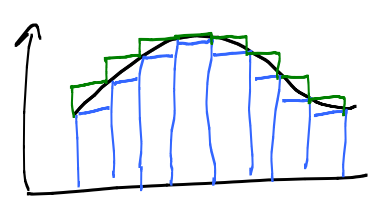
Lemma $\downarrow$ $\qquad\qquad$ $\downarrow$

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_u \, d\theta}_{\frac{\beta-\alpha-\varepsilon}{2\pi}} \qquad\qquad \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_o(e^{in\gamma})}_{\frac{\beta-\alpha+\varepsilon}{2\pi}}$$

Finally, let $\varepsilon \to 0$ to see that $\frac{1}{N}\sum_{1}^{N} \chi(e^{in\gamma}) \to \frac{\beta-\alpha}{2\pi}$.

Hmmm: Lemma holds for $f =$ "step functions."

Can continue this idea to show lemma holds for Riemann integrable fns.



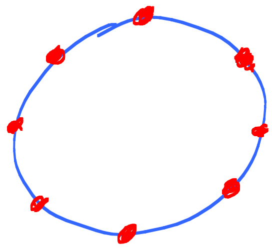
How to generate random numbers in $[0,1]$.

Get two gigantic primes p and q. Take a big N.

"Throw a dart": $\underbrace{N \cdot \frac{p}{q} \mod \mathbb{Z}}_{= \text{decimal part}}$

What about rational multiples of π? $e^{in\frac{p}{q}\pi}$

$\frac{\pi}{4}$



repeats!