Lecture 28 Laplace Transform $\rightsquigarrow$ Fourier Transform

Fourier Transform: $$\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$f(x) \overset{\text{hope}}{=} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$$

$$\approx \sum_{n=-\infty}^{\infty} \hat{f}(w_n)\, e^{isw_n}\, \Delta w_n$$

think: expand f in terms <u>all</u> basic freq.

<u>Laplace Transform:</u>

$$\mathcal{L}[f(x)](s) = \int_0^{\infty} f(x)\, e^{-sx}\, dx = F(s)$$

Makes sense for $s > K$ when f is piecewise C^1 and "of exponential type": $|f(x)| \le M e^{Kx}$ for some $M > 0$, $K > 0$.

<u>Big facts:</u> $\mathcal{L}$ is a linear operator.

1) $$\boxed{\mathcal{L}[f'] = sF(s) - f(0)} \qquad F = \mathcal{L}[f]$$

$$\boxed{\mathcal{L}[f''] = s^2 F(s) - s f(0) - f'(0)}$$

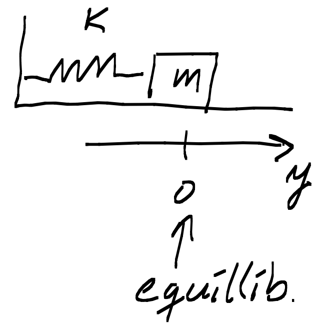
2) If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > K$ (for some K), then $f_1 \equiv f_2$ on $[0, \infty)$.

Consequently, $\mathcal{L}^{-1}$ makes sense!

Plan: $\mathcal{L}[\underbrace{\text{linear ODE}}_{y=f(x)}] = \text{Algebraic eqn in } F$

Solve for F. $\quad \text{Sol}^n \; f = \mathcal{L}^{-1}[F]$.

EX: $\underset{m=1}{\uparrow}\, y'' + \underset{\underbrace{c=5}_{\text{coeff of friction}}}{5}\, y' + \underset{K=6}{6}\, y = \underbrace{e^{-4t}}_{\text{external force } E}$



$$\text{Force} = m\,a$$

$$\underbrace{-Ky}_{\text{Hooke's}} - \underbrace{cy'}_{\text{friction}} + E = my''$$

Initial Conditions.

$$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases}$$

Sol^n: Hit ODE with $\mathcal{L}$:

$$\underbrace{\mathcal{L}[y''] + 5\mathcal{L}[y'] + 6\mathcal{L}[y]} = \mathcal{L}[e^{-4t}] = \int_0^\infty e^{-4t} e^{-st}\,dt$$

$$\left[s^2 Y(s) - s\cdot\underset{y(0)}{1} - \underset{y'(0)}{2}\right] + 5\left[sY(s) - \underset{y(0)}{1}\right] + 6Y(s)$$

$$= \int_0^\infty e^{-(s+4)t}\,dt$$

$$= \lim_{B\to\infty}\left[\frac{-1}{s+4} e^{-(s+4)t}\right]_0^B$$

$$= 0 - \left(\frac{-1}{s+4}\right) = \frac{1}{s+4}$$

when $s > -4$

$$(s^2+5s+6)Y = 7 + s + \frac{1}{s+4}$$

$$Y(s) = \underbrace{\frac{s+7}{s^2+5s+6}}_{H_1(s)} + \underbrace{\frac{1}{(s^2+5s+6)(s+4)}}_{H_2(s)}$$

Finally, get sol^n y by "undoing" Lap. Trans.!

How:
$$\frac{s+7}{s^2+5s+6} = \frac{s+7}{(s+2)(s+3)} \overset{\uparrow}{=} \frac{A}{s+2} + \frac{B}{s+3} = \mathcal{L}\Big[\underbrace{Ae^{-2t}+Be^{-3t}}_{\text{a sol}^n\text{ to homog. eqn!}}\Big]$$

partial frac's

$$\frac{1}{(s^2+5s+6)(s+4)} = \frac{1}{(s+2)(s+3)(s+4)} = \frac{a}{s+2} + \frac{b}{s+3} + \frac{c}{s+4}$$

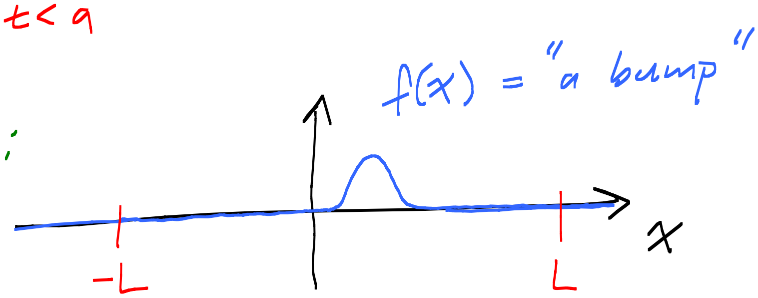
$$= \mathcal{L}\Big[\underbrace{ae^{-2t} + be^{-3t}}_{\text{solves homog.}} + \underbrace{ce^{-4t}}_{\text{particular sol}^n\text{ to non-homog eqn}}\Big]$$

Engineers love it:

$$\mathcal{L}\big[\underbrace{u_a(t)}_{=\begin{cases}1 & t\geq a\\ 0 & t<a\end{cases}} f(t-a)\big] = e^{-as}F(s)$$

$F=\mathcal{L}[f]$

Motivation for Fourier integral:



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

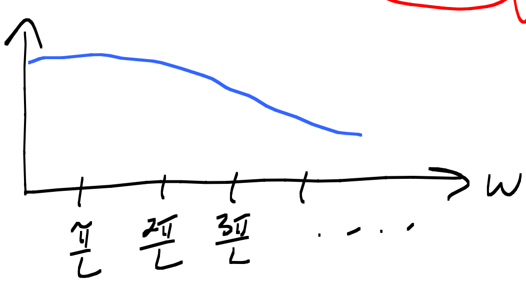
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \underbrace{\frac{1}{2L}\int_{-L}^{L} f(t)\,dt}_{\to 0 \text{ as } L \to \infty} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\underbrace{\frac{1\cdot\pi}{L}}_{\Delta w}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt\right)\cos\underbrace{\frac{n\pi}{L}}\cdot x \qquad w_n = \frac{n\pi}{L}$$

$$+ \sum \ldots\ldots \sin \ldots \sin \ldots$$

$$\underset{\text{Riemann sum}}{=} \lim_{L\to\infty}\sum_{1}^{\infty}\frac{1}{\pi}\underbrace{\left(\int_{-\infty}^{\infty}\underbrace{f(t)}_{\substack{\text{zero}\\ \text{outside}\\ \text{bump.}}}\cos w_n t\,dt\right)}_{A(w_n)}\cos w_n x \underbrace{\Delta w}_{\frac{\pi}{L}}$$

∞ ok here



$$f(x) = \int_0^{\infty}\left(A(w)\cos wx + B(w)\sin wx\right)dw$$

where

$$\begin{cases} A(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\cos wt\,dt \\[2ex] B(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\sin wt\,dt \end{cases}$$

Fact: If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty}|f|\,dx$ is finite, then f "=" its Fourier integral. (= at continuous places; = midpoint of jumps)