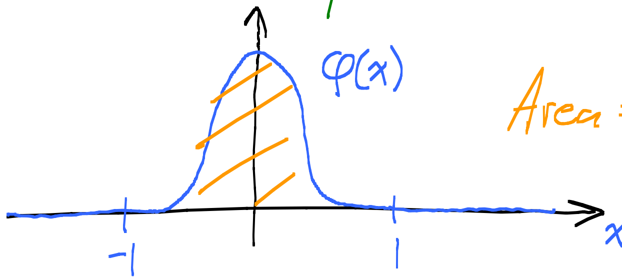
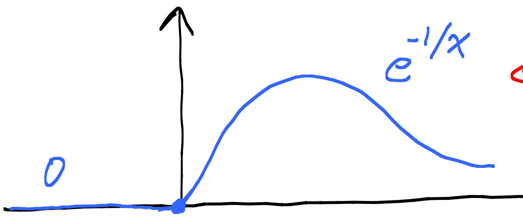
Lecture 33 Applications of bump functions

C_0^∞ bump fcn



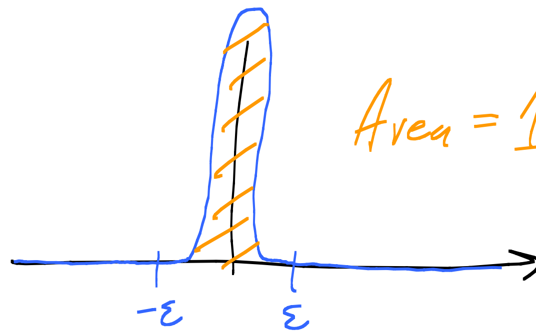
Area = 1, symmetric, 0 outside $[-1/2, 1/2]$



$e^{-1/x}$ ← easier than e^{-1/x^2} from last time

C^∞ smooth around origin

Define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$



Area = 1

Convolution: $$(f*g)(x) = \int_{-\infty}^{\infty} f(\underbrace{x-t}_{\tau})\, g(t)\, dt$$

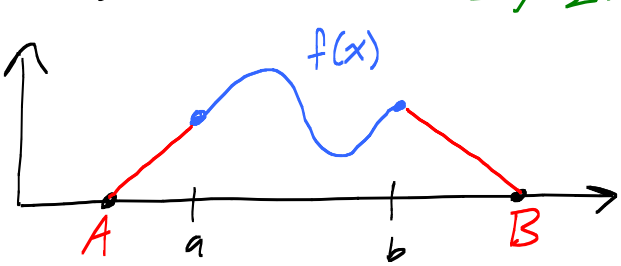
$$= \int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau$$ ← commutes

when $f, g \in \mathcal{M}(\mathbb{R})$.

Thm: Given a continuous fcn f on $[a,b]$ and $\varepsilon_0 > 0$, there is a C^∞-smooth fcn on $[a,b]$ with

$|f(x) - f_\varepsilon(x)| < \varepsilon_0$ for all $x \in [a,b]$.

Pf:



extend f to $\mathbb{R}$ to be continuous &

have compact support.

__Claim:__ $\varphi_\varepsilon * f \to f$ uniformly on $[a,b]$.

__Why:__ $\underbrace{(\varphi_\varepsilon * f)(x)}_{= f * \varphi_\varepsilon} - \underbrace{f(x)}_{= f(x) \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\,dt}_{=1}}$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\left[f(x-t) - f(x)\right] dt$$

$$|\,\text{eee}\,| \leq \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t)\,\underbrace{\left|f(x-t) - f(x)\right|}\,dt$$

f cont on a closed interval $[A,B]$.
So f is uniformly cont there.
Can choose ε such that
$|f(x) - f(y)| < \varepsilon_0$ when $|x-y| < \varepsilon$

$y = x - t$

$$\leq \underbrace{\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t)\,\varepsilon_0\,dt}_{= \varepsilon_0 \cdot 1.} \quad \text{when } \varepsilon \text{ small enough}$$

__Last step:__ Show $\varphi_\varepsilon * f$ is C^∞-smooth:

$$DQ = \frac{(\varphi_\varepsilon * f)(x+h) - (\varphi_\varepsilon * f)(x)}{h}$$

$$= \int_{\cancel{-\infty}\ A}^{\cancel{\infty}\ B} \underbrace{\frac{\varphi_\varepsilon((x+h)-t) - \varphi_\varepsilon(x-t)}{h}}_{dq} f(t)\, dt$$

$$dq \approx \varphi_\varepsilon'(x-t)$$

$$\parallel$$

$$\varphi_\varepsilon'(\tilde{\imath}), \quad \tilde{\imath} \text{ beteen } x-t \text{ and } (x-t)+h$$

MVT: $\dfrac{\varphi(b)-\varphi(a)}{b-a} = \varphi'(\tilde{\imath})$ for some $\tilde{\imath} \in (a,b)$

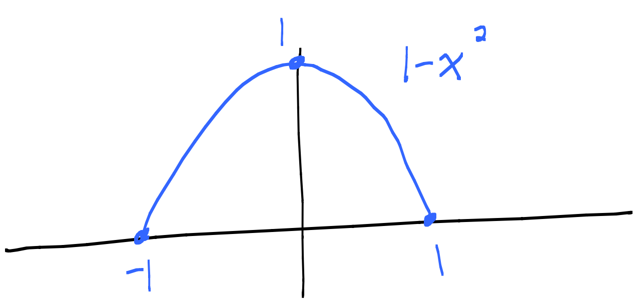
Aha! $\varphi_\varepsilon'(\tilde{\imath}) - \varphi_\varepsilon'(x-t)$ is small when h is small because φ_ε' is continuous. And <u>uniformly</u> small

Repeat for higher derivatives.

<u>Thm</u>: $\dfrac{d^n}{dx^n}(\varphi_\varepsilon * f) = \varphi_\varepsilon^{(n)} * f$

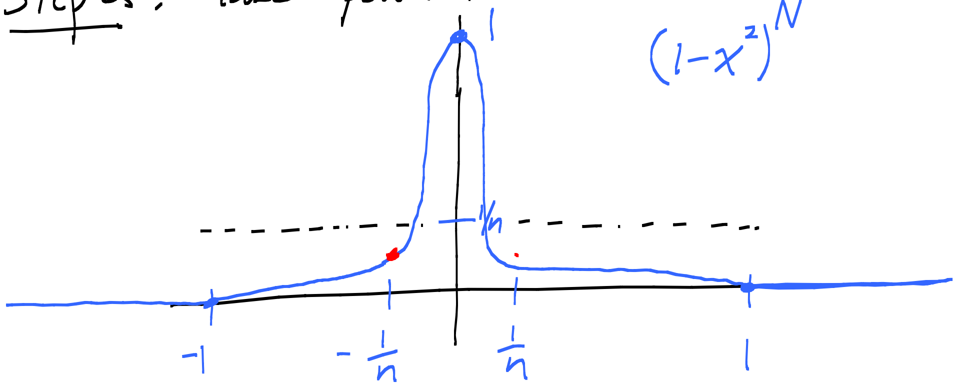
<u>Weierstraß Thm</u>: A polynomial bump fcn will make $\varphi_\varepsilon * f$ a poly!

<u>Pf</u>: <u>Step 1</u>:



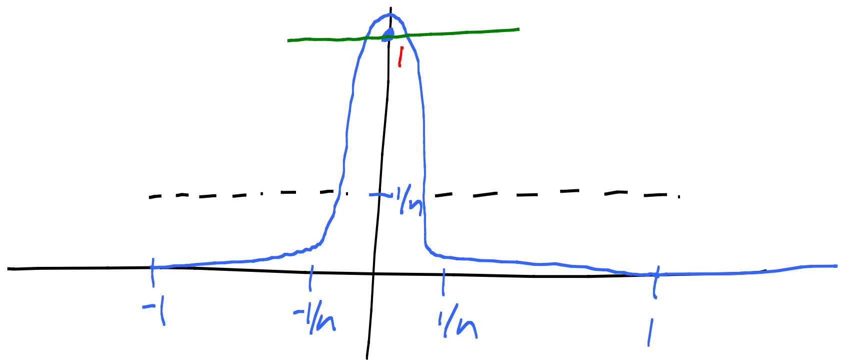
Step 2: Take powers.

$(1-x^2)^N$



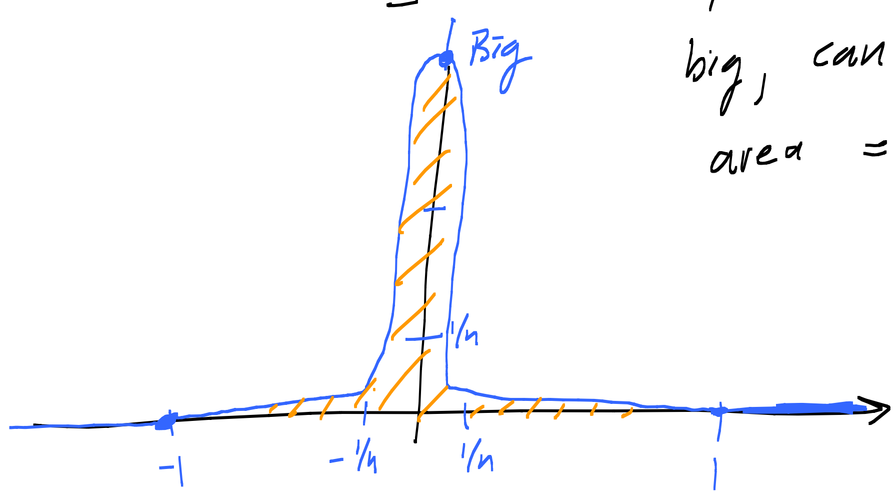
Fact: $0 < r < 1$.
$r^N \to 0$
as $N \to \infty$

Step 3: $(1+\varepsilon)(1-x^2)^N$



Step 4: $\left[(1+\varepsilon)(1-x^2)^N\right]^M$

By taking M big, can make area $= A > 1$



Define $Q_n(x) = \frac{1}{A}\left[(1+\varepsilon)(1-x^2)^N\right]^M$

Pf above shows $Q_n * f \to f$ uniformly as $n \to \infty$.

Claim: $Q_n * f$ is a poly!

$$(Q_n * f)(x) = \int_{-\infty}^{\infty} \underbrace{Q_n(x-t)}_{\sum_0^m a_n (x-t)^n} f(t)\, dt$$

$$= \sum_{0 \le k, \ell \le m} A_{k\ell}\, x^k t^\ell$$

$$= \sum A_{k\ell}\, x^k \int_{\cancel{-\infty}\ A}^{\cancel{\infty}\ B} t^\ell f(t)\, dt$$

How to feel about bump fcns

$$\underbrace{\varphi_\varepsilon * f}_{} \longrightarrow f$$

$\approx$ Riemann sum

$$\sum \underbrace{\varphi_\varepsilon(x - t_n)}_{\substack{\text{unit mass} \\ \text{near } t_n}} f(t_n)\, \Delta t \longleftarrow$$ linear combination of bump fcns translated.

Feeling: Fourier inversion true on bumps

$\Longrightarrow$ true for $f \approx \sum$ bumps.

Fantastic fact that makes life easier: e^{-x^2} makes a great bump fcn and $\widehat{\left(e^{-x^2}\right)} = c\, e^{-x^2}$