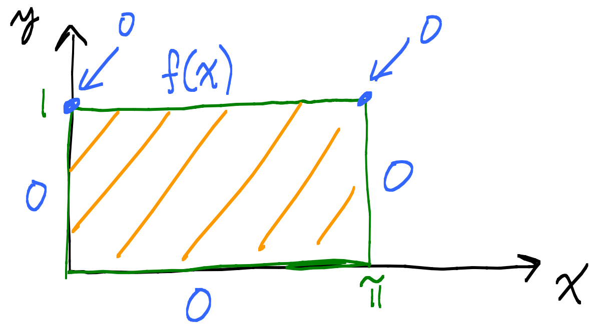
Lecture 39 Dirichlet problem on a rectangle

Review problems for final exam on home page



Find steady state temp, given fixed temp on edges.

Math prob: $\underbrace{\frac{\partial u}{\partial t}}_{\to 0 \text{ as } t\to\infty} = c^2 \Delta x$: Laplace's eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

BC $\begin{cases} 1.\ u(x,0) = 0, & 0 \le x \le \pi \\ 2.\ u(\pi, y) = 0, & 0 \le y \le 1 \\ 3.\ u(0,y) = 0, & 0 \le y \le 1 \end{cases}$ plus 4. $u(x,1) = f(x) \leftarrow$ given

Method of separation of variables: Try $u(x,y) = X(x)Y(y)$

PDE: $X''Y + XY'' \overset{\text{want}}{=} 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

BC's 2, 3 $\begin{cases} u(\pi, y) = X(\pi)Y(y) \overset{\text{want}}{=} 0 \\ u(0,y) = X(0)Y(y) = 0 \end{cases}$

Need $X(0) = 0$, $X(\pi) = 0$.

Famous prob: $\boxed{\begin{array}{l} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{array}}$

Only has non-zero sol'ns when $\lambda = -n^2$:

Get $X_n(x) = \sin nx$

Back to Y prob:

$$-\frac{Y''}{Y} = -n^2$$

$$\boxed{\begin{array}{c} Y'' - n^2 Y = 0 \\ Y(0) = 0 \end{array}}$$

BC #1

$$u(x,0) = 0$$

$$X(x)\underbrace{Y(0)} = 0$$

need $Y(0) = 0$

$$Y(y) = c_1 e^{ny} + c_2 e^{-ny}$$

want

$$Y(0) = c_1 + c_2 = 0$$

$$\boxed{c_2 = -c_1}$$

So $Y(y) = c_1 e^{ny} - c_1 e^{-ny} = \underbrace{2c_1}\left(\frac{e^{ny} - e^{-ny}}{2}\right)$

Take $c_1 = \frac{1}{2}$

Get $Y_n(y) = \sinh ny$ and

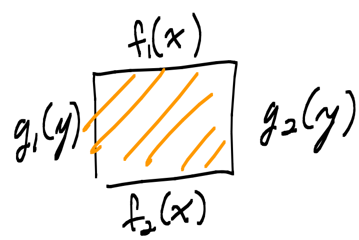
$$u_n(x,y) = X_n(x)\,Y_n(y) = \sin nx \sinh ny$$

Finally, try $u(x,y) = \sum_{n=1}^{\infty} A_n \sin nx \sinh ny$.

Last BC: #4 $u(x,1) = \sum_{n=1}^{\infty} A_n \sin nx \sinh n \cdot \overset{\text{want}}{1} = f(x)$

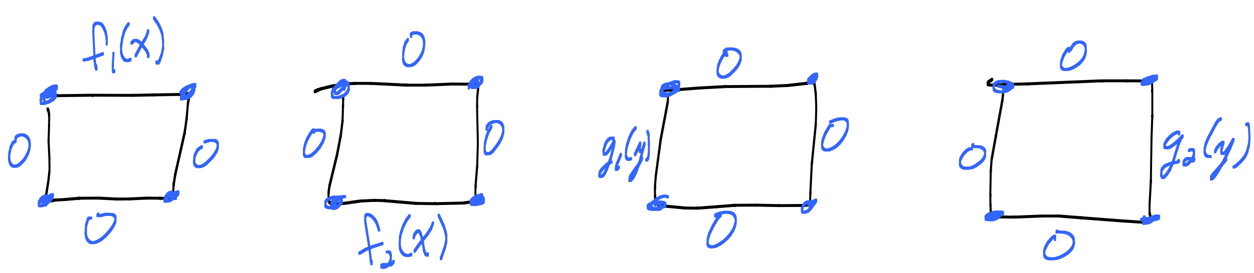
$$= \sum_{n=1}^{\infty} \underbrace{\left(A_n \sinh n \right)}_{\text{Fourier sine series coeff for } f(x)!} \sin nx$$

$$\boxed{A_n = \frac{1}{\sinh n} \cdot \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx}$$

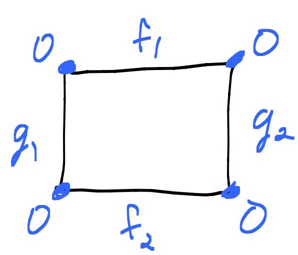
what about general prob 

?

Do 4 problems:



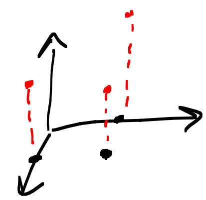
Add up the four sol^{n}s.



<u>Problem</u>: Can I cook a harmonic fcn with given values at the 4 corners of a rectangle.

Hmmm: $ax + by + c$ is harmonic

3 pts determine a plane



Need more harmonic fcns.

$$r^n \cos n\theta, \quad r^n \sin n\theta \quad \text{are all harmonic.}$$

In fact, they are harmonic poly's:

$$r^n \cos n\theta = \operatorname{Re}\Big[\underbrace{r^n \cos n\theta + i\, r^n \sin n\theta}\Big]$$

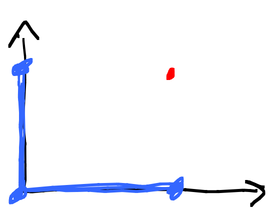
$$= r^n(\cos n\theta + i \sin n\theta)$$

$$= r^n (e^{i\theta})^n = (re^{i\theta})^n$$

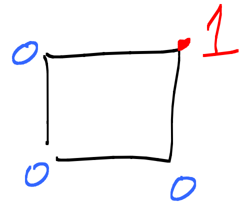
$$= (x+iy)^n$$

Hmmm. $(x+iy)^2 = (x^2-y^2) + i(2xy)$

harmonic

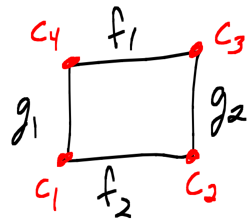


xy is harmonic

Get harmonic fcn 

$\frac{1}{\pi} xy$

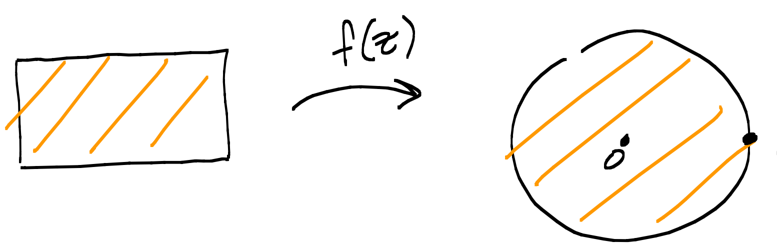
Aha! Get a linear fcn $ax + by + c$ with proper values at blue corners. Fix 4th pt using $\frac{1}{\pi} xy$.

General prob: 

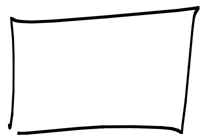
Subtract a harmonic fcn h with values c_j at 4 corners.

$$u - h = (\text{Sum of 4 sol}^{\text{n}}\text{s cooked up above})$$

What's the Poisson kernel for a rectangle?



Riemann mapping thm: $\exists$ 1-1, onto, ~~analytic~~ conformal, map of rectangle to unit disc. Analytic fcns preserve harmonic fcns. Can "pull-back" the Poisson kernel for disc to get Poisson kernel for the rectangle — involves elliptic fcns.



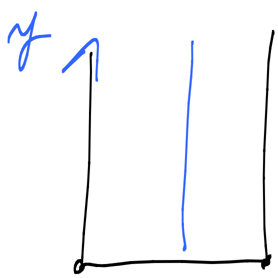
Fourier series

Laplace transform in x

Fourier Sine or Cosine series in x

Fourier Transform (no boundary)

real or complex



Take a transform in y var.