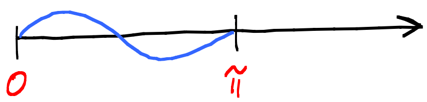
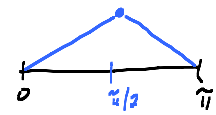
Lesson 2 Sol^n of the harpsichord problem



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

B.C. $u(0,t) \equiv 0$
$u(\pi,t) \equiv 0$

I.C. $u(x,0) = f(x) =$ 

$\frac{\partial u}{\partial t}(x,0) = g(x) \leftarrow$ take $\equiv 0$ for "pluck"

Johann Bernoulli's sol^n: Try $u(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

$$\boxed{\begin{array}{l} X'' - \lambda X = 0 \\ \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{array}}$$

$$T'' - \lambda T$$

Case $\lambda = 0$: $X(x) = c_1 x + c_2$. B.C.'s $\Rightarrow X(x) \equiv 0$.

Only get $u(x,t) = X(x)T(t) \equiv 0$.

Case $\lambda > 0$: Only get zero sol^n.

Case $\lambda < 0$: Write $\lambda = -k^2$. $r^2 + k^2 = 0$ roots $r = \pm ki$

$e^{ikx} = \underbrace{\cos kx} + i \underbrace{\sin kx}$ is a complex sol^n.

get two real sol^{n}s from one comp. sol^n.

$X(x) = \cancel{c_1 \cos kx} + c_2 \sin kx \leftarrow$ Wronskian $= \det \begin{bmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{bmatrix}$

$= k \cdot 1 \neq 0$

Next: B.C.s:

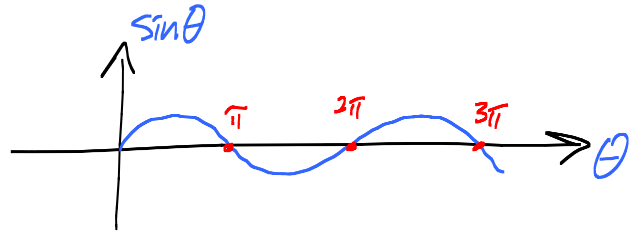
$X(0) = c_1 \cdot 1 + c_2 \cdot 0 \overset{\text{want}}{=} 0$

Get $\boxed{c_1 = 0}$

B.C. #2 : $X(\pi) = c_2 \sin k\pi = 0$ ← want

Don't want $c_2 = 0$ because we'd have the zero sol^n again.

Need $\boxed{\sin k\pi = 0}$



Need $k\pi = n\pi$ for $n = 1, 2, 3, \ldots$

$\boxed{k = n}$ ← $\lambda = -k^2 = -n^2$ $\quad n = 1, 2, 3, \ldots$

Get $X_n(x) = c_2 \sin nx$ $\qquad \lambda = -n^2$

take $c_2 = 1$.

Back to T prob: $T'' - \underset{-n^2}{\lambda} T = 0$

$$T(t) = A_n \cos nt + B_n \sin nt$$

Get $u_n(x,t) = \sin nx \left(A_n \cos nt + B_n \sin nt \right)$

PDE ✓
BC ✓
IC No way!

Aha! PDE is linear. So linear combos of sol^n's are sol^n's.
BC is "homogeneous". Linear combos satisfy BC's.

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(A_n \cos nt + B_n \sin nt \right)$$

Finally, wory about I.C.

$$\begin{cases} u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \overset{\text{want}}{=} f(x) \Leftarrow \text{pluck shape} \\ \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin nx \left(-A_n n \cdot 0 + B_n n \cdot 1\right) \overset{\text{want}}{=} 0 \end{cases}$$

Aha! Take all B_n's $= 0$,

Sol^n ?
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt}$$

Need
$$\sum_{n=1}^{\infty} A_n \sin nx = f(x)$$

Fourier sine series : Key $\{\sin nx\}_{n=1}^{\infty}$ are "orthogonal" on $[0,\pi]$:

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$n=m$
$$\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2nx) \, dx = \frac{\pi}{2} \checkmark$$

Idea: $\left[\vec{V} = c_1 \vec{u}_1 + \cdots + c_m \vec{u}_m + \cdots + c_k \vec{u}_k\right] \cdot \vec{u}_m \leftarrow \perp$

$$\vec{v}\cdot\vec{u}_m = c_1 \underbrace{\vec{u}_1\cdot\vec{u}_m}_{0} + \cdots + c_m \underbrace{\vec{u}_m\cdot\vec{u}_m}_{\text{not zero}} + \cdots c_k \underbrace{\vec{u}_k\cdot\vec{u}_m}_{0}$$

$$c_m = \frac{\vec{v}\cdot\vec{u}_m}{\vec{u}_m\cdot\vec{u}_m}$$

want

$$f(x)\cdot \sin mx = \left(\sum_{n=1}^{\infty} A_n \sin nx\right)\cdot \sin mx$$

$$\int_0^{\pi} f(x)\cdot \sin mx\,dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi} \sin nx \cdot \sin mx\,dx}_{\substack{\text{zero if } n\neq m \\ =\frac{\pi}{2} \text{ if } n=m}}$$

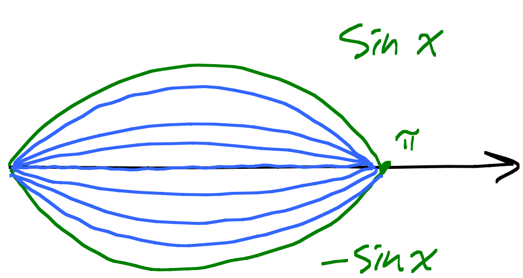
$$= A_m\cdot\frac{\pi}{2}$$

$$\boxed{A_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin mx\,dx}$$

change m to n.

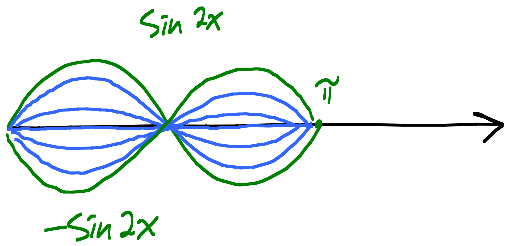
<u>Modes of oscillation</u> : $u_n(x,t) = \sin nx \cos nt$

$n=1$



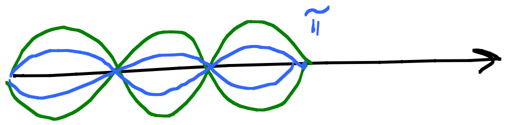
$\sin x \cos t$

$n=2$

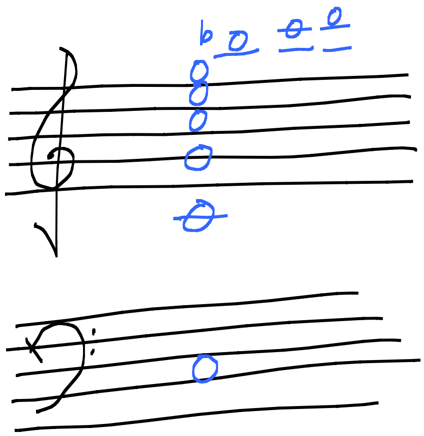


$\sin 2x \cos 2t$

$n=3$



$\sin 3x \cos 3t$



$n=5$ E
$n=4$ C
$n=3$ G
$n=2$ C

$n=1$ C