Lesson 5 Complex notation for Fourier series

Complex exponential $e^z = e^{x+iy} = e^x e^{iy} \overset{\text{def}^n}{=} \underbrace{\left(e^x \cos y\right) + i\left(e^x \sin y\right)}_{= E(x+iy) \text{ today only}}$

Euler: $e^{iy} = \cos y + i \sin y$

Facts about $E(z)$: 0) $E(0) = 1$

1) $E(z)$ is complex diff'ble: $\lim_{z \to a} \frac{E(z)-E(a)}{z-a}$ exists (call it $E'(z)$)

and $E'(z) = E(z)$.

2) $E(z+w) = E(z) \cdot E(w)$

Pf: $E(z_1)E(z_2) = e^{x_1}\left(\cos y_1 + i \sin y_1\right) e^{x_2}\left(\cos y_2 + i \sin y_2\right)$

$$= \underbrace{e^{x_1} e^{x_2}}_{e^{x_1+x_2}} \Big[\underbrace{\left(\cos y_1 \cos y_2 \underset{\uparrow\, i^2}{-} \sin y_1 \sin y_2\right)}_{\cos(y_1+y_2)} + i \underbrace{\left(\cos y_1 \sin y_2 + \sin y_1 \cos y_2\right)}_{\sin(y_1+y_2)} \Big]$$

$= E(z_1 + z_2)$ ✓

2') $\underbrace{E(z-z)}_{E(0)=1} = E(z)E(-z) \leftarrow$ $E(z)$ is never 0

$E(-z) = 1/E(z)$

2'') $E(z-w) = E(z)/E(w)$

3) $E(x) = e^x$ when $x \in \mathbb{R}$.

Fun facts: $\left.\begin{array}{l} E(0) = 1 \\ E'(z) = E(z) \end{array}\right\}$ determine $E(z)$

$E(z)$ is the unique $\mathbb{C}$-diff'ble fcn extenting e^x to $\mathbb{C}$.

4) $E(i\theta) = \cos\theta + i\sin\theta$

2 plus 4 yield <u>DeMoivre's formula</u> : $(e^{i\theta})^N = e^{iN\theta}$

$$(e^{i\theta})^N = e^{i\theta} \cdot e^{i\theta} \cdots e^{i\theta} = e^{iN\theta} \checkmark$$

True for negative $N \in \mathbb{Z}$ too! (2').

5) $e^{i\theta} = \cos\theta + i\sin\theta$

$+ \quad e^{-i\theta} = \cos\theta - i\sin\theta \qquad -$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad\qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

<u>Important function</u>: Dirichlet Kernel

$$D_N(t) = \sum_{n=-N}^{N} e^{int} = \sum_{n=1}^{N} e^{-int} + \sum_{n=0}^{N} e^{int}$$

$$= \sum_{n=1}^{N} (\cos nt - i\sin nt) + \sum_{n=0}^{N} (\cos nt + i\sin nt)$$

$$= 1 + 2\sum_{n=1}^{N} \cos nt = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}$$

<u>Why</u>: $1 + \cos t + \cos 2t + \cdots + \cos Nt$

$$= \mathrm{Re}\left[1 + e^{it} + e^{i2t} + \cdots + e^{iNt}\right]$$

$$= \mathrm{Re}\left[1 + (e^{it}) + (e^{it})^2 + \cdots + (e^{it})^N\right]$$

Aha! Geometric series!

$$S_N = 1 + z + z^2 + \cdots + z^N$$

$$- \quad z S_N = \quad z + z^2 + \cdots + z^N + z^{N+1}$$

$$(1-z) S_N = 1 - z^{N+1}$$

$$S_N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z} \longrightarrow \frac{1}{1-z} \text{ if } |z|<1$$

to continue $= \mathrm{Re}\left[\dfrac{1-(e^{it})^{N+1}}{1-(e^{it})}\right]$ $\quad z = e^{it}$

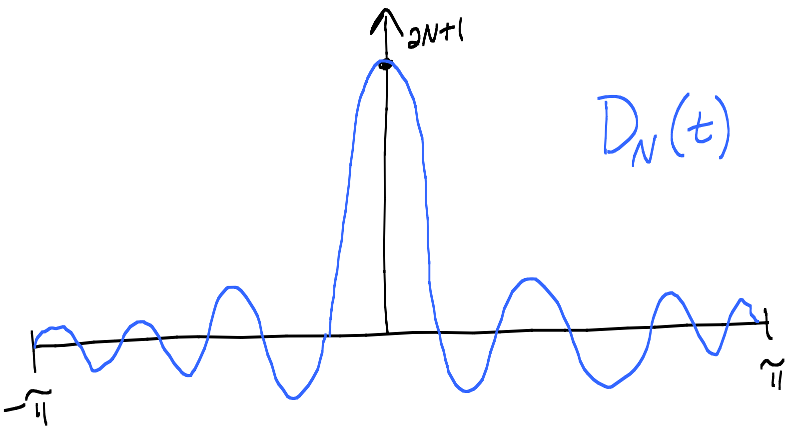
$$= \mathrm{Re}\left[\frac{1 - e^{i(N+1)t}}{1 - e^{it}} \cdot \frac{e^{-it/2}}{e^{-it/2}} \cdot \frac{\left(\frac{-1}{2i}\right)}{\left(\frac{-1}{2i}\right)}\right]$$

$$= \mathrm{Re}\left[\frac{\frac{i}{2} e^{-it/2} - \frac{i}{2} e^{i(N+\frac{1}{2}t)}}{\sin \frac{t}{2}}\right]$$

$$= \frac{\frac{1}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\left(N+\frac{1}{2}\right)t}{\sin\frac{t}{2}} = \frac{1}{2} + \frac{1}{2}\,\frac{\sin\left(N+\frac{1}{2}\right)t}{\sin t/2}$$

$$D_N(t) = 1 + 2\sum_{n=1}^{N} \cos nt = 2 \cdot \mathrm{Re}\, d - 1 = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}$$

L'Hôpital's: $$\lim_{t\to 0} D_N(t) = \lim_{t\to 0} \frac{(N+\frac{1}{2})\cos(N+\frac{1}{2})t}{\frac{1}{2}\cos t/2} = 2N+1$$ Big!



Key property: $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$$

Because of $+1$ term.

$$\int_{-\pi}^{\pi} \cos nt\, dt = 0$$

$n = 1, 2, \ldots, N$

Complex version of Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \underbrace{\cos nx}_{\frac{e^{inx}+e^{-inx}}{2}} + b_n \underbrace{\sin nx}_{\frac{e^{inx}-e^{-inx}}{2i}}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\underbrace{\underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{\frac{1}{2}(a_n - ib_n)}}_{c_n} e^{inx} + \underbrace{\underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{\frac{1}{2}(a_n + ib_n)}}_{c_{-n}} e^{-inx}\right]$$

$$c_n = \frac{1}{2}(a_n - i\,b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\underbrace{\left[\cos nx - i \sin nx\right]}_{e^{-inx}}\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Got $\quad c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

Same formula for c_{-n} !

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\underbrace{1}_{\substack{\shortparallel \\ e^{-i\cdot 0}}}\,dx \qquad ✓$$