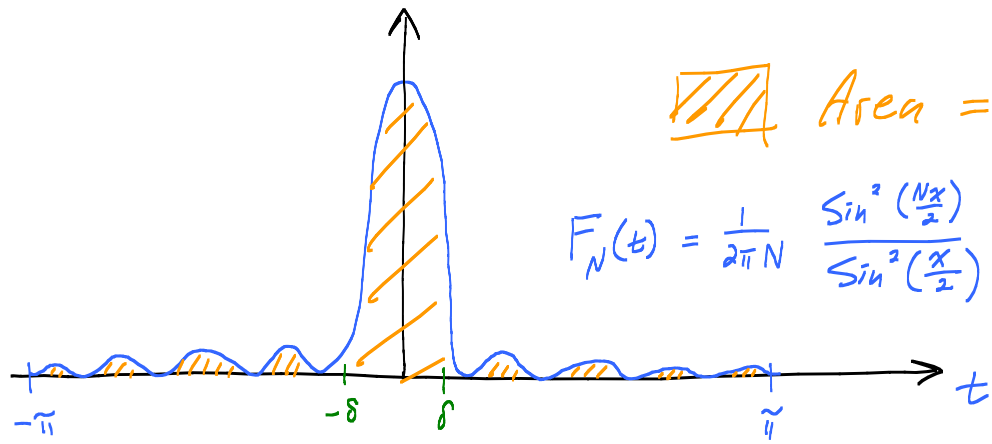
Lesson 10 The Fejér kernel and Fejér's theorem



Area = 1

$$F_N(t) = \frac{1}{2\pi N} \frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} \geq 0$$

$\longrightarrow 0$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ as $N \to \infty$

$$F_N(t) = \frac{D_0(t) + \cdots + D_{N-1}(t)}{N}$$

where
$$D_N(t) = \frac{1}{2\pi} \sum_{k=-N}^{N} e^{ikt}$$

$$= \frac{1}{2\pi}\left[\underbrace{\sum_{k=1}^{N} e^{-ikt}} + \underbrace{\sum_{k=0}^{N} e^{ikt}}_{\frac{1-e^{i(N+1)t}}{1-e^{it}}} \right]$$

$$\sum_{k=1}^{N} \left(e^{-it}\right)^k$$

$$\underbrace{e^{-it}}_{\frac{1}{e^{it}}} \underbrace{\left(\sum_{k=0}^{N-1} \left(e^{-it}\right)^k \right)}_{\frac{1-\left(e^{-it}\right)^{(N-1)+1}}{1-e^{-it}}}$$

$$\frac{1}{e^{it}} \cdot \frac{1-\left(e^{-it}\right)^{(N-1)+1}}{1-e^{-it}}$$

$$\frac{1-e^{-iNt}}{e^{it}-1}$$

$$\boxed{D_N(t) = \frac{1}{2\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{1-e^{it}}}$$

$$F_N = \frac{D_0 + \cdots + D_{N-1}}{N}$$

$$2\pi N F_N = \sum_{n=0}^{N-1} \frac{e^{-int} - e^{i(n+1)t}}{1 - e^{it}}$$

$$= \frac{1}{1-e^{it}} \left[\sum_{n=0}^{N-1} \left(e^{-it}\right)^n - \underbrace{e^{it}}_{\frac{1}{e^{-it}}} \sum_{n=0}^{N-1} \left(e^{it}\right)^n \right]$$

$$= \frac{1}{1-e^{it}} \left[\frac{1-e^{-iNt}}{1-e^{-it}} - \frac{1}{e^{-it}} \cdot \frac{1-e^{iNt}}{1-e^{it}} \right]$$

$$= \frac{1}{1-e^{it}} \left[\frac{1-e^{-iNt}}{1-e^{-it}} + \frac{1-e^{iNt}}{1-e^{-it}} \right]$$

$$= \frac{2 - e^{-iNt} - e^{iNt}}{(1-e^{it})(1-e^{-it})}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{\overbrace{2 - e^{-iNt} - e^{iNt}}^{2-2\cos Nt}}{\underbrace{2 - e^{it} - e^{-it}}_{2-2\cos t}}$$

Aha!

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right)$$

$$Nt = 2\theta$$

$$\theta = \frac{Nt}{2}$$

$$= \frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}$$

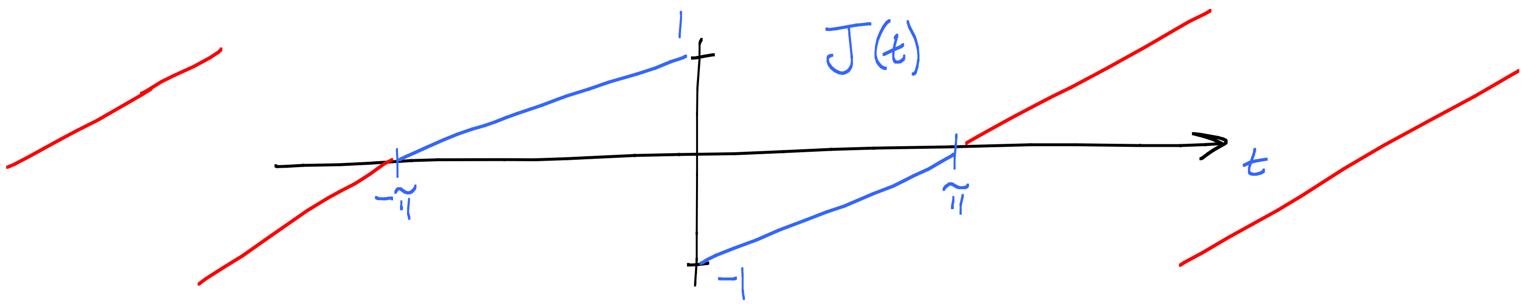
Yes!

$$\left(1 + z + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}\right)$$

How do we know Fourier Series converge to midpoint of jumps?

At the moment, we know F.S. $\to f(x)$ for f piecewise continuous at pts x where f is C^1-smooth nearby (or merely Lipschitz at x).

Quintessential jump fcn:

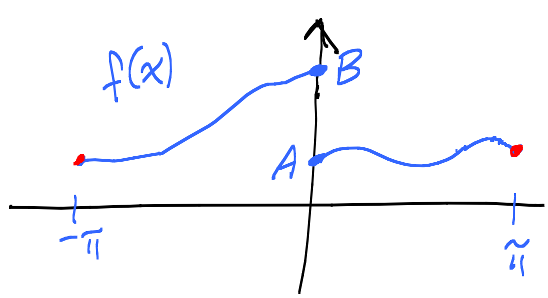


2π-periodic extension. J is odd. So all cosine terms are zero and $J(x) = \sum_{n=1}^{\infty} B_n \sin nx$.

At jump, $\Sigma = 0$, the midpoint of the jump!

$\Sigma \to J$ at smooth parts, including $\pm\pi$.

Big idea:



2π periodic, C^1 except at jump.

Aha! $\quad f(x) - \dfrac{(B-A)}{2} J(x) = F(x)$ $\quad$ Cancels the jump!

F might have a kink at $x=0$, but no jump.

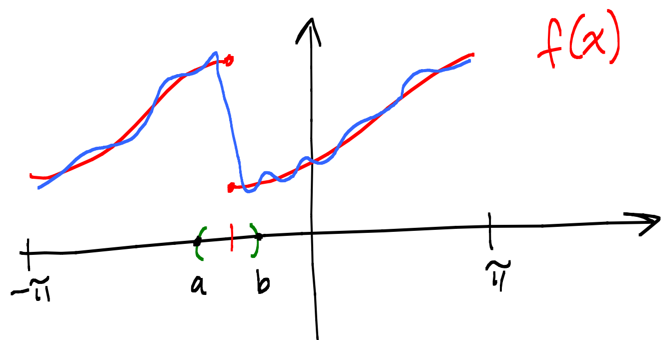
Fourier series is linear operation.

$$F = f - \frac{(B-A)}{2} J$$

$$\begin{pmatrix} \text{F.S. for} \\ F \end{pmatrix} = \begin{pmatrix} \text{F.S for} \\ f \end{pmatrix} - \frac{(B-A)}{2} \begin{pmatrix} \text{F.S. for} \\ J \end{pmatrix}$$

Conclude: F.S. for $f \to$ Midpoint of the jump at 0.

L^2 convergence:



$$\int (\text{Red} - \text{blue})^2 dx = \underbrace{\int_{-\pi}^{a} (<\varepsilon)^2 dx}_{\varepsilon^2 \cdot 2\pi} + \int_a^b \underbrace{(\text{Jump} + \varepsilon)^2}_{<M} dx$$

$$+ \underbrace{\int_b^{\pi} (<\varepsilon)^2 dx}_{\varepsilon^2 \cdot 2\pi}$$

$$< \varepsilon^2 4\pi + M \underbrace{(b-a)}_{\text{can make as small as I want.}}$$

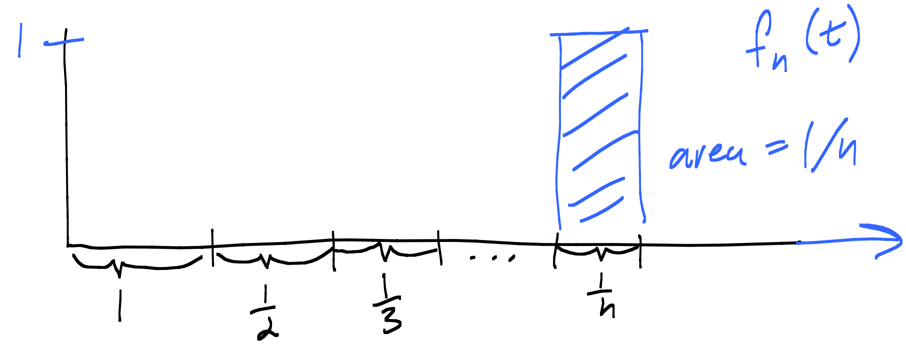
Fact: Can find trig polys T_n such that the L^2-norm

$$\| f - T_n \|^2 = \int_{-\pi}^{\pi} |f(x) - T_n(x)|^2 \, dx \to 0 \text{ as } n \to \infty$$

even when f is piecewise continuous. (Jumps).

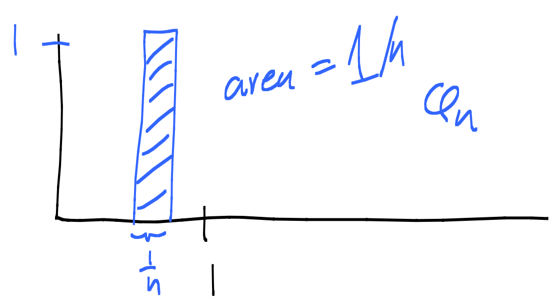
Cor: Parseval's identity holds for fcns with jumps.

EX:



Shift mod-1.

$$\int_0^1 \varphi_n \, dx = \frac{1}{n}$$



$\varphi_n \to 0$ in L^1-norm, but $\varphi_n \not\to 0$ pointwise!