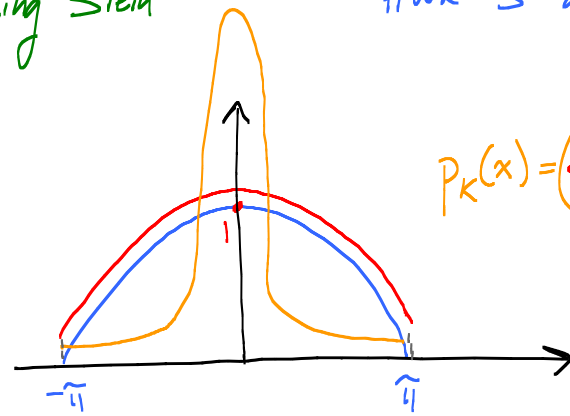
Lesson 12 Reading Stein

HWK 3 due Wed.

Stein p. 40:



$p_K(x) = (\varepsilon + \cos x)^K$

$\cos x$

Stein: $p_K(x)$ is a trig poly!

$$T_N = a_0 + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta)$$
$$= \sum_{n=-N}^{N} c_n e^{in\theta}$$

Fact: poly $P(\cos\theta, \sin\theta)$

$$= P\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right) = \cdots \text{ expand} \cdots, \text{ simplify}$$

$(e^{i\theta}\cdot e^{i\theta}\cdot e^{i\theta}\cdot e^{-i\theta} = e^{i2\theta}, \text{etc.})$

$= $ trig poly!

Stein: $\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z} \iff f \perp$ all trig polys

$$\hat{f}(n) = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \overline{e^{inx}}\, dx$$

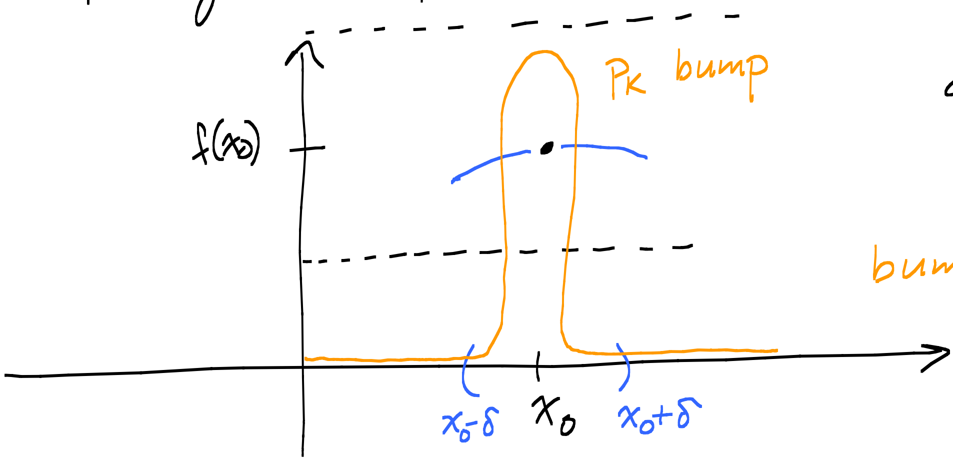
$$= \frac{1}{2\pi}\langle f, e^{inx}\rangle$$

these generate the trig polys.

Stein: Suppose f is continuous, $\hat{f}(n) = 0 \;\; \forall n \in \mathbb{Z}$

$\Longrightarrow f \equiv 0.$ [Bell: Parseval's gives this easily.]

Pf: Suppose $f(x_0) \neq 0$. May assume $f(x_0) > 0$, by replacing f by $-f$ if not.



$\varepsilon = \dfrac{f(x_0)}{2}$

See $\underbrace{\int_{-\pi}^{\pi} P_K f \, dx}_{= 0 \text{ if } f \perp \text{trig polys}} > 0 \qquad \lightning.$

How to guess true formulas for Fourier series:

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$f'(x) \sim \sum_{-\infty}^{\infty} in c_n e^{inx}$$

True that $\widehat{f'}(n) = in \hat{f}(n)$ via formula for $\widehat{f'}(n)$ and integration by parts.

$$\underbrace{f(x-a)}_{T_a f} \sim \sum_{-\infty}^{\infty} c_n e^{in(x-a)} = \sum_{-\infty}^{\infty} e^{-ina} c_n e^{inx}$$

Is it true that $\widehat{T_a f}(n) = e^{-ina}\hat{f}(n)$?

Yes, it is via change of var. $\tilde{x} = x - a$ in formula.

Suppose $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$

$$T_a f = \lim_{N\to\infty} T_a S_N \leftarrow$$ If you understand convergence at $x=0$, its the same at $x=a$.

<u>Thm</u>: F.S. $\rightarrow$ midpoints of jumps.

Also clear from $$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t)\, \underbrace{\left[f(x-t) - f(x)\right]}_{\text{what works at } x=0 \text{, works for } x\neq 0.}\, dt$$

<u>Dirichlet problem on unit disc</u>: Given a <u>continuous</u> fcn $\varphi(\theta)$ on the unit circle, find a continuous fcn u on $\{(r,\theta) : 0 \le r \le 1, -\pi \le \theta \le \pi\} = \overline{D_1(0)}$ that is harmonic on the interior $D_1(0) = \{(r,\theta) : r < 1\}$ such that $u(1,\theta) = \varphi(\theta) \leftarrow \varphi(-\pi) = \varphi(\pi)$.

Bernoulli's idea of "separation of variables"

Try $u(r,\theta) = R(r)\,\Theta(\theta)$.

Harmonic: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underbrace{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{\text{polar form}} \equiv 0$

Laplace operator or Laplacian

Want $\quad \frac{\partial^2}{\partial r^2} R\Theta + \frac{1}{r}\frac{\partial}{\partial r} R\Theta + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} R\Theta \overset{\text{want}}{=} 0$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Mult by r^2, divide by $R\Theta$: $\quad \underbrace{\frac{r^2R'' + rR'}{R}}_{\text{fcn of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fcn of } \Theta} = \lambda$

They both must be constant λ!

$\underline{\Theta \text{ problem}}$: Periodic bndry conditions! A) $\Theta(-\pi) = \Theta(\pi)$

B) $\Theta'(-\pi) = \Theta'(\pi)$

$$\Theta'' + \lambda\Theta = 0$$

(A) no rips!

(B) no creases!

Need "wiggling" sol'ns to make periodic B.C. hold.

$\underline{\text{Case } \lambda = 0}$: $\quad \Theta(\theta) = A\theta + B \qquad A(-\pi) + B = A\pi + B \quad (1)$

$\Theta'(\theta) = A \qquad$ ✓ $\quad (2)$

(1) implies $A = 0$.

Get a solution! $\Theta(\theta) = \text{const.}$, take it to be 1.

Case $\lambda < 0$: $\lambda = -k^2$ $\quad \Theta'' - k^2 \Theta = 0$

$$\Theta(\theta) = A \cosh k\theta + B \sinh k\theta$$

Impossible to hook up B.C.'s! Only get zero sol^n.

Case $\lambda > 0$: $\lambda = k^2$ $\quad \Theta(\theta) = A \cos k\theta + B \sin k\theta$

Play around. Can only make B.C.'s hold

when k = positive integer.

Get $\Theta_n(\theta) = a_n \cos nx + b_n \sin nx$

$\lambda = 0$: $\Theta_0(\theta) = a_0$

Next time: $R(r)$ problem : $\dfrac{r^2 R'' + r R'}{R} = \lambda = n^2$

$r^2 R'' + r R - n^2 R = 0 \leftarrow$ Euler eqn