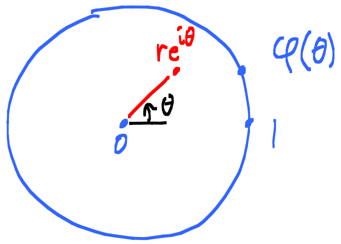
Lesson 13 Solution to the Dirichlet problem on the unit disc in $\mathbb{R}^2$

Formula sheet on Home page

HWK 4 due Wed.



Looking for $u(r,\theta)$ continuous on $\overline{D_1(0)}$, harmonic on $D_1(0)$, with $u(1,\theta) = \varphi(\theta)$.

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Try $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Θ problem: $\Theta'' + \lambda\,\Theta = 0$ periodic boundary cond:

$$\begin{cases} \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta'(\pi) \end{cases}$$

$\lambda = 0$: Got $\Theta_0(\theta) \equiv a_0$

$\lambda < 0$: No periodic sol^{n}s except zero sol^n.

$\lambda > 0$: Get periodic sol^{n}s exactly when $\lambda = n^2$, $n = 1, 2, 3, \ldots$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

Back to $R(r)$ problem: $$\frac{r^2 R'' + r R'}{R} = \lambda$$

Case $\lambda = 0$: $\quad r^2 R'' + r R' = 0$

$$R'' + \frac{1}{r} R' = 0 \qquad \text{Let } V = R'(r)$$

$$\boxed{\frac{dV}{dr} + \frac{1}{r} V = 0}$$

linear first order ODE

$$\text{Int. factor: } \quad u = e^{\int \frac{1}{r} dr} = e^{\ln r} = r$$

$$r \cdot \text{ODE} = \underbrace{r \frac{dV}{dr} + V}_{\frac{d}{dr}[uV]} = 0$$

$$\frac{d}{dr}[uV] \equiv 0$$

know temp $= CQ(\theta)$ on boundary

Steady state temp is harmonic!

$$rV = \text{const.},\ c$$

$$V = \frac{c_1}{r}$$

$$\text{So} \quad R' = \frac{c_1}{r} \quad \text{and}$$

$$R = \int \frac{c_1}{r} dr = \underbrace{c_1 \ln r}_{\text{dump this part.}} + c_2$$

dump this part. Take $c_1 = 0$.

$$\text{Get } \quad R_0(r) = c_2. \quad \leftarrow \text{Take } c_2 = 1.$$

$$u_0(r,\theta) = \Theta_0(\theta) R_0(r) = a_0 \cdot c_2 = a_0$$

Case $\lambda = n^2$: $\quad \dfrac{r^2 R'' + r R'}{R} = \lambda = n^2$

$$r^2 R'' + r R' - n^2 R = 0 \quad \leftarrow \text{Euler eqn.}$$

Guess $R(r) = r^p$

$$R'(r) = p r^{p-1}$$

$$R''(r) = p(p-1) r^{p-2}$$

Plug in $\quad r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p \overset{\text{want}}{=} 0$

$$\Big[\underbrace{p(p-1) + p - n^2}_{p^2 - n^2} \Big] r^p = 0$$

need $= 0 \qquad p = \pm n$

Two! Good! $\quad R(r) = c_1 r^n + \underbrace{c_2 r^{-n}}_{\text{discard this part!}}$

Take $R_n(r) = r^n$

$$u_n(r,\theta) = R_n \Theta_n = r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Finally, since Δ is a linear operator, try

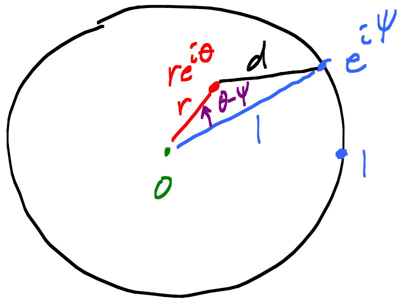
$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

Hope: This sums to a harmonic fcn.

Last thing: Want $\quad u(1,\theta) = \underbrace{a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)}_{\text{Fourier series!}} \overset{\text{want}}{=} \varphi(\theta)$

Getting Poisson formula and kernel :

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{1-r^2}{1+r^2-2r\cos(\theta-\psi)}}_{d^2 \text{ by law of cosines}} \varphi(\psi)\, d\psi$$



Poisson kernel $\quad P(r,\theta,\psi) = \frac{1}{2\pi}\,\frac{1-r^2}{|re^{i\theta}-e^{i\psi}|^2}$

Good kernel :

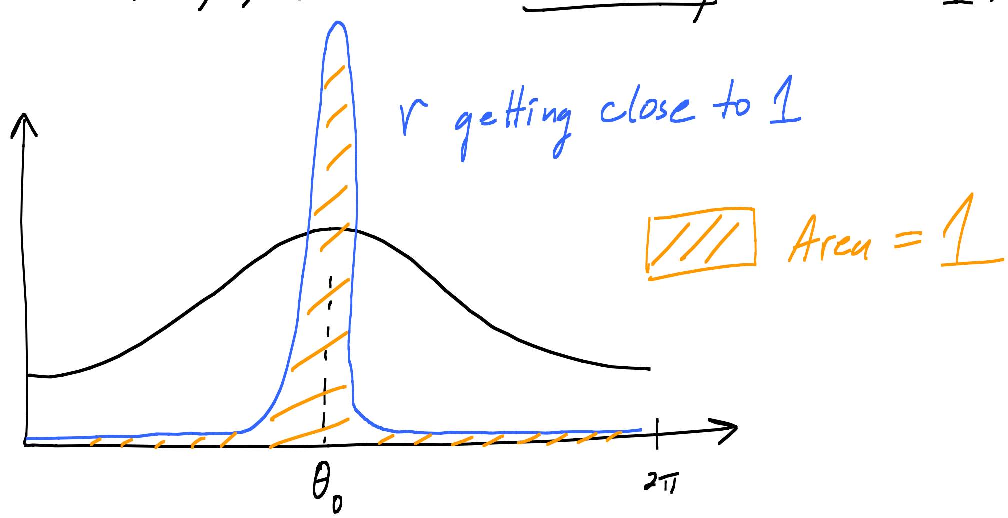
1) $P \geq 0$

2) $\int_0^{2\pi} P(r,\theta,\psi)\, d\psi = 1$

3) For ψ outside $(\theta_0-\alpha, \theta_0+\alpha)$,

$P(r,\theta_0,\psi) \to 0$ uniformly as $r \to 1$.

Picture:



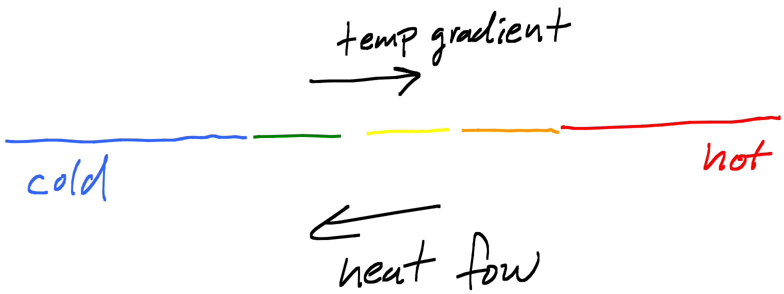
4) $\Delta_{(r,\theta)} P(r,\theta,\Psi) \equiv 0$

Dirichet problem is the steady state heat sol^n:

Heat eqn : $\underbrace{\frac{\partial u}{\partial t}}_{\substack{\equiv 0 \\ \text{at steady state}}} = c\,\Delta u$

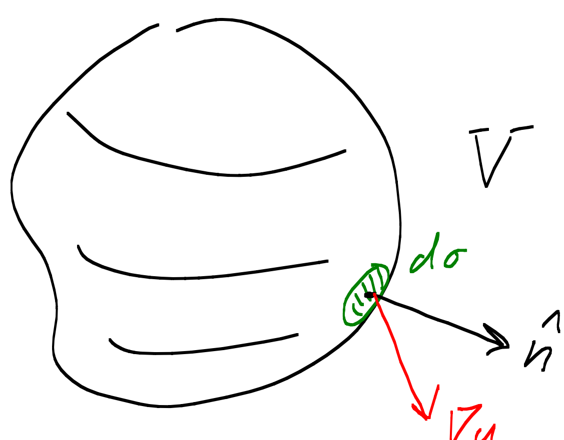
Get Laplace eqn : $\Delta u \equiv 0.$ ← u harmonic

Kelvin's law: (Heat flow) is proportional to (temp gradient)



Const of prop is negative!

Control volume: In $\mathbb{R}^3$:



One calorie is the amount of heat needed to raise cm^3 of water 1 degree Kelvin.

Heat content of V is $\iiint_V u \, dV$

Rate of change $\frac{d}{dt}\left(\iiint_V u \, dV\right)$

But Kelvin's law shows rate of change is also =

$$c \iint_{bV} \nabla u \cdot \hat{n} \, d\sigma$$