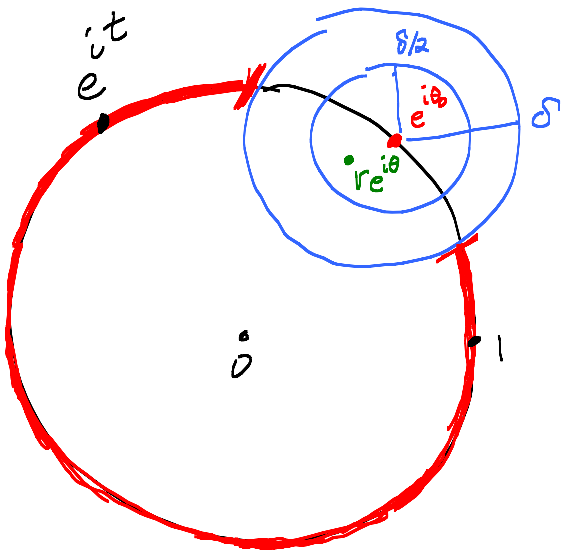
Lesson 15 The Poisson integral formula & harmonic fcns.

HWK 4 due Wed.

3)



$I_\delta = \{t : |e^{it} - e^{i\theta_0}| > \delta\}$

$$P(r,\theta,t) = \frac{1-r^2}{|re^{i\theta} - e^{it}|^2}$$

$$\leq \frac{1-r^2}{\left(\frac{\delta}{2}\right)^2} \leftarrow (1-r)\underbrace{(1+r)}_{<2}$$

$$< \frac{8(1-r)}{\delta^2} \longrightarrow 0 \text{ as } r \nearrow 1$$

1) $P(r,\theta,t) > 0$

2) $\displaystyle\int_0^{2\pi} P(r,\theta,t)\,dt = 1$ for r, θ, $0 < r < 1$.

4) $\Delta_{(r,\theta)} P(r,\theta,t) \equiv 0$

$$u(r,\theta) = \int_0^{2\pi} \underbrace{\frac{1}{2\pi}\frac{1-r^2}{|re^{i\theta} - e^{it}|^2}}_{P(r,\theta,t)} f(t)\,dt \quad \text{solves D. Prob:}$$

Continuous on $\overline{D_1(0)}$, harmonic on $D_1(0)$,

$u(1,\theta) = f(\theta)$.

<u>Easy part</u>: OK to take $\Delta_{(r,\theta)}$ under $\int$ sign.

Prop 4 shows u harmonic inside $D_1(0)$.

Interesting part: $u(r,\theta)$ is continuous up to unit circle

and $\lim\limits_{\underbrace{re^{i\theta}}_{\text{inside } D_{\delta/2}(e^{i\theta_0})} \to e^{i\theta_0}} u(r,\theta) = f(\theta_0)$

$$u(r,\theta) - f(\theta_0) = u(r,\theta) - \underbrace{\left(\int_0^{2\pi} P(r,\theta,t)\,dt\right)}_{=1 \text{ by } (2)} f(\theta_0)$$

$$\int_0^{2\pi} P(r,\theta,t)\left[f(t) - f(\theta_0)\right] dt$$

Aha! $\underbrace{\int_{t\in I_\delta}}_{P \text{ small}} + \underbrace{\int_{\theta_0-\delta}^{\theta_0+\delta}}_{f \text{ part small}}$

Let $\varepsilon > 0$.
f continuous at θ_0.
So $\exists\ \delta > 0$ such that $|f(t) - f(\theta_0)| < \frac{\varepsilon}{2}$ when $|t-\theta_0| < \delta$.

f continuous on $[0,2\pi] \Rightarrow |f|$ bounded by M there.

So $|f(t) - f(\theta_0)| < |f(t)| + |f(\theta_0)| < 2M$.

$$|u(r,\theta) - f(\theta_0)| \le \int_{t\in I_\delta} \underbrace{P(r,\theta,t)}_{< \frac{1}{2\pi}\frac{8(1-r)}{\delta^2}} \underbrace{|f(t) - f(\theta_0)|}_{2M}\,dt$$

$$< \frac{1}{2\pi}\frac{8(1-r)}{\delta^2} \cdot 2M \, (\underbrace{\text{length } I_\delta}_{< 2\pi})$$

Can make $< \frac{\varepsilon}{2}$ as $r \nearrow 1$
with $re^{i\theta} \in D_{\delta/2}(e^{i\theta_0})$

$$+ \int_{\theta_0-\delta}^{\theta_0+\delta} \underbrace{P(r,\theta,t)}_{>0} \underbrace{\left| f(t) - f(\theta_0) \right|}_{<\frac{\varepsilon}{2}} \, dt$$

$$< \frac{\varepsilon}{2} \underbrace{\int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta,t)\, dt}_{< \int_0^{2\pi} P(r,\theta,t)\,dt \;=\; 1}$$

$$< \frac{\varepsilon}{2}$$

See $u(r,\theta) \to f(\theta_0)$ as $re^{i\theta} \to e^{i\theta_0}$
inside $D_{\delta/2}(e^{i\theta_0})$.

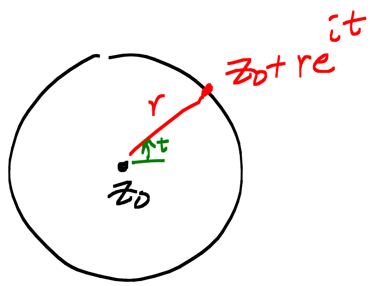
This famous proof is "Schwarz's theorem."

Hmmm. Take $r=0$ in Poisson formula.

$$P(0,\theta,t) = \frac{1-0^2}{|0\cdot e^{i\theta} - e^{it}|^2} \equiv 1$$

Aha! $\quad u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt$

Fact: Harmonic fcns satisfy the averaging property



$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt$$

Why: Green's identities can be used to show this.

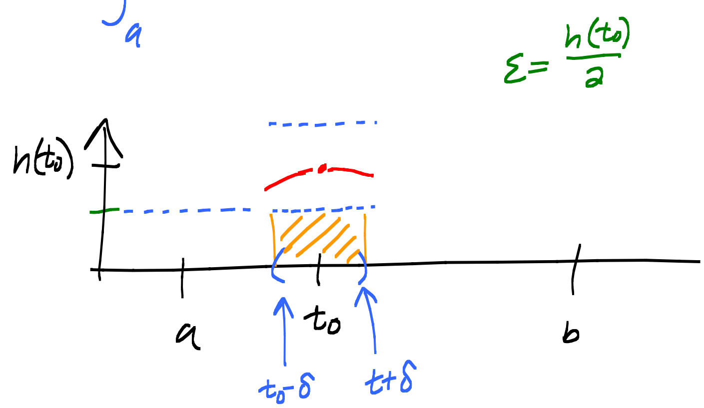
Fact: Ave property $\Longrightarrow$ Max principle.

Calculus lemma: Suppose $h(t) \geq 0$ continuous on $[a,b]$.

Then $\int_a^b h(t)\,dt = 0 \Longrightarrow h \equiv 0$.

$\varepsilon = \frac{h(t_0)}{2}$

Pf:



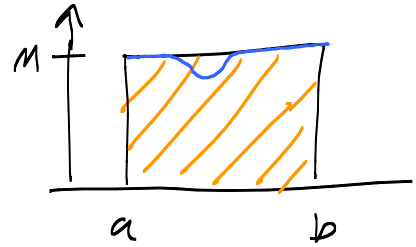
Suppose $h(t_0) \neq 0$. Can assume $h(t_0) > 0$. otherwise replace h by $-h$.

See $\int_a^b h(t)\,dt > 2\delta\varepsilon > 0$ $\lightning$

Lemma: If $h(t)$ is continuous on $[a,b]$ and $h(t) \leq M$,

then $\int_a^b h(t)\,dt = M(b-a) \Longrightarrow h(t) \equiv M$.

Pf: Apply first lemma to $M - h(t)$.



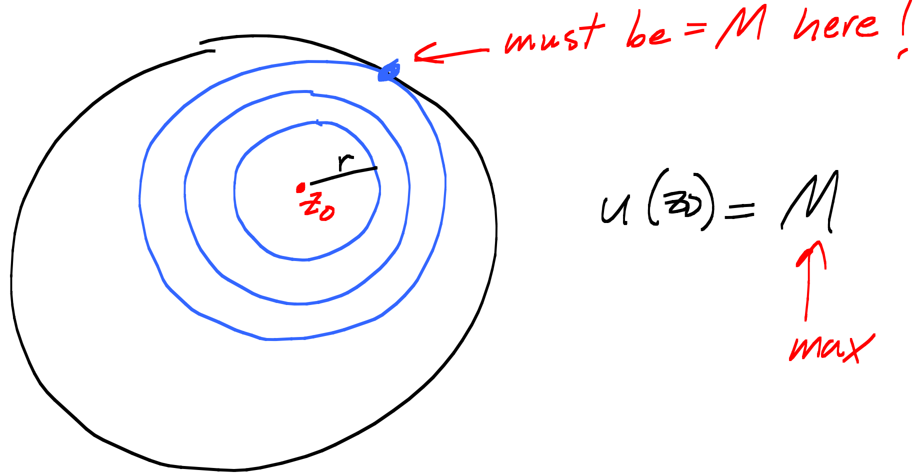
Max Princ: Suppose u is harmonic on $D_1(0)$ and continuous up to $\overline{D_1(0)}$. Then u attains its max on the boundary.

Pf: u continuous on $\overline{D_1(0)}$, closed and bounded.

So u assumes its max M at a pt $z_0 \in \overline{D_1(0)}$.

Case z_0 in boundary. ✓ u attains max on bndry.

Case $z_0 \in D_1(0)$.



$u(z_0) = M$
max

$$\underbrace{u(z_0)}_{M} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{\leq M}\, dt \leq \frac{1}{2\pi} M \cdot 2\pi = M$$

Calculus lemma $\Rightarrow$ $u(z_0 + re^{it}) \equiv M$ all t.

Let $r \nearrow$ biggest possible disc touching bndry.

Second version of max princ. If a harmonic fcn assumes a local max at $z_0 \in D_1(0)$, then it must be const.

Big application of max princ: Sol^n to D. Prob is unique.

Why: Two sol^{n}s u_1, u_2 with $f(t)$ boundary values, then

$u_1 - u_2 = 0$ on bndry! Max princ

$\Rightarrow u_1 - u_2 \le 0$ on $\overline{D_1(0)}$.

Repeat for $u_2 - u_1$. Get $u_2 - u_1 \le 0$.

So $u_1 \equiv u_2$ on $\overline{D_1(0)}$.

Another biggy: Harmonic fcns are C^∞-smooth!

Why: u is C^2-smooth and $\Delta u \equiv 0$.

$u =$ Poisson integral of $u(e^{it})$

can diff'tiate under $\int$ with respect to r and θ at will.