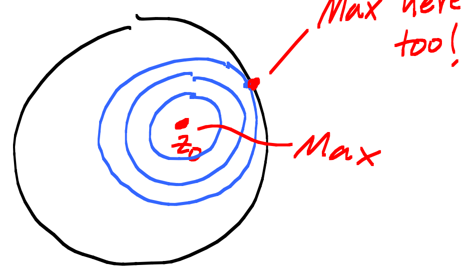
# Lesson 16 Harmonic functions

Harmonic functions satisfy the Averaging property:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\, dt$$

Theorem: If $u$ is continuous and satisfies the ave property, then $u$ is harmonic (and so $C^\infty$-smooth!).

Pf: Ave property $\Rightarrow$ max princ.




Clever trick. Have $u$ satisfying ave prop. Let $$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta,t)\,\underbrace{u(1,t)}_{\text{boundary values of } u.}\,dt$$

Know $U$ harmonic in $D_1(0)$, continuous on $\overline{D_1(0)}$ and $U(1,\theta) = u(1,\theta)$. $U$ harmonic $\Rightarrow$ $U$ satisfies ave prop.

Aha! $u - U$ is a continuous fcn that satisfies ave prop.

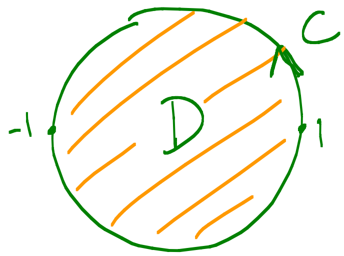
So Max princ holds $u - U \le \max_{C_1(0)} \underbrace{(u-U)}_{0 \text{ on } C_1(0)} \le 0.$

Repeat for $U - u$ and get $U - u \le 0$.

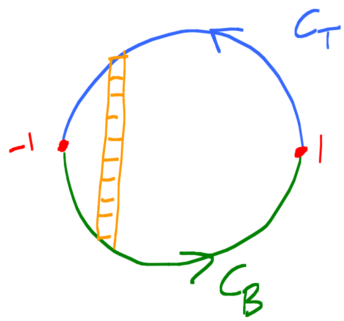
So $u \equiv U \leftarrow$ harmonic ✓

## Importance of Green's theorem and Green's identities.

Green's theorem easy and elementary on a disc!




$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$




$$C_B: \quad x(t) = t, \qquad -1 \le t \le 1$$
$$y(t) = -\sqrt{1-t^2} = h_B(t)$$

$$-C_T: \quad x(t) = t, \qquad -1 \le t \le 1$$
$$y(t) = \sqrt{1-t^2} = h_T(t)$$

Claim: $\displaystyle\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dx\,dy$

$$\int_C = \int_{C_B} + \int_{C_T} = \left(\int_{C_B} - \int_{-C_T}\right) P\,dx$$

$$= \int_{-1}^{1} P(t, h_B(t))\,\underbrace{1 \cdot dt}_{\dot{x}(t)\,dt} - \int_{-1}^{1} P(t, h_T(t))\,1 \cdot dt$$

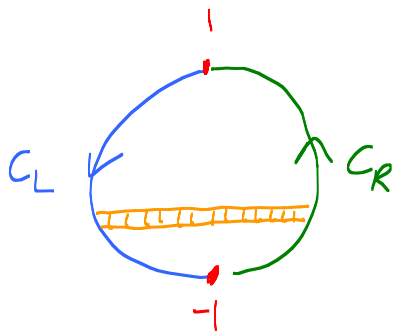
$$= -\int_{-1}^{1} \underbrace{\Big[P(t, h_T(t)) - P(t, h_B(t))\Big]}_{\int_{h_B(t)}^{h_T(t)} \frac{\partial}{\partial y} P(t,y)\,dy}\,dt$$

Fund Thm Calc!

Change $t$ to $x$: $\displaystyle = -\int_{x=-1}^{1} \left( \int_{y=h_B(x)}^{h_T(x)} \frac{\partial P}{\partial y}(x,y)\,dy \right) dx$

$$= -\iint_D \frac{\partial P}{\partial y}\, dx\, dy \quad \checkmark$$

## The other half:




$C_R$: $\quad y(t) = t \qquad -1 \le t \le 1$

$\qquad x(t) = \sqrt{1-t^2} = g_R(t)$

$-C_L$: $\quad y(t) = t \qquad -1 \le t \le 1$

$\qquad x(t) = -\sqrt{1-t^2} = g_L(t)$

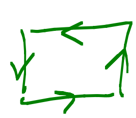
$$\int_C = \left( \int_{C_R} - \int_{-C_L} \right) Q\, dy$$

$$= \int_{-1}^{1} Q(g_R(t), t)\, \underbrace{1 \cdot dt}_{\dot{y}(t)\,dt} - \int_{-1}^{1} Q(g_L(t), t)\, dt$$

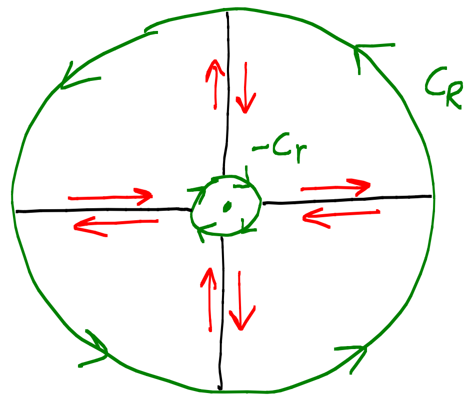
$$= \int_{-1}^{1} \left( \int_{g_L(t)}^{g_R(t)} \frac{\partial Q}{\partial x}(x, t)\, dx \right) dt$$

$$= \iint_D \frac{\partial Q}{\partial x}\, dx\, dy \quad \checkmark$$

Important thing: Having top, bottom, left, right.

Rectangles also easy. 

Cool thing:




Add up Green's Thm on four pieces of pie. Cuts cancel!

Get $\left(\int_{C_R} + \int_{-C_r}\right) P\,dx + Q\,dy = \iint_{A(r,R)} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

Green's 2nd identity:

$$\int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \iint_D (u\Delta v - v\Delta u)\,dx\,dy$$

Take disc with hole at origin. $u =$ harmonic fcn.

$v = \operatorname{Ln} r \leftarrow$ harmonic!

Note: $v \equiv 0$ on $r = 1$.

$$\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = \frac{1}{r} = 1 \text{ on } r = 1$$

$$\left(\int_{C_1} + \int_{-C_\varepsilon}\right) u\underbrace{\frac{\partial v}{\partial n}}_{=1 \text{ on } C_1} - \underbrace{v}_{=0 \text{ on } C_1}\frac{\partial u}{\partial n}\,ds = \iint_{A(\varepsilon,1)} \underbrace{u\Delta v - v\Delta u}_{=0}\,dx\,dy$$

$$= \int_{C_1} u\,ds + \int_{-C_\varepsilon} u\cdot\frac{1}{r}\underbrace{ds}_{r\,d\theta} - \operatorname{Ln} r\cdot\frac{\partial u}{\partial n}\underbrace{ds}_{r\,d\theta}$$

Let $\varepsilon \to 0$. Use $r \ln r \to 0$ as $r \to 0$.

Get $$\int_{C_1} u\, ds = \lim_{\varepsilon \to 0} \int_0^{2\pi} u(\varepsilon e^{it})\, dt = 2\pi\, u(0).$$

$\ln r$ is the Green's fcn. $\leftarrow$ Harmonic on $D$, zero on boundary, blows up like $\ln r$ at origin.

Green's theorem $\Rightarrow$ Divergence theorem:

$$\int_C \vec{F} \cdot \hat{n}\, ds = \iint_D \operatorname{Div} \vec{F}\, dx\, dy$$

$\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath}$ $\qquad$ $\operatorname{Div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$