Lesson 18 Hot wire problem

Initial condition: $u(x,0) = f(x)$

$u(x,t) =$ temp at x at time t

Boundary conditions:

$$\begin{cases} u(0,t) = 0 & \text{all } t \\ u(\pi,t) = 0 & \text{all } t \end{cases}$$

Heat eqn: $\dfrac{\partial u}{\partial t} = c\,\dfrac{\partial^2 u}{\partial x^2}$ Take $c = 1$.

Try to find special sol^{n}s: $u(x,t) = X(x)T(t)$

Plug into PDE: $\dfrac{\partial}{\partial t}(XT) = \dfrac{\partial^2}{\partial x^2}(XT)$

$$X(x)T'(t) = X''(x)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda, \text{ const.}$$

T problem: $T' - \lambda T = 0$

$$T(t) = Ce^{\lambda t}$$

X prob: $X'' - \lambda X = 0$

B.C. $u(0,t) = X(0)\underbrace{T(t)}_{} = 0$

don't want $T(t) \equiv 0$.

So need $X(0) = 0$

$u(\pi,t) = X(\pi)T(t) = 0$

need $X(\pi) = 0$

Same Sturm-Liouville prob as string problem:

$$X'' - \lambda X = 0$$

$$\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

<u>Case $\lambda = 0$</u>: $X(x) = Ax + B$. BC $\Rightarrow$ $X(x) \equiv 0$.

<u>Case $\lambda > 0$</u>: $X(x) = a \cosh\sqrt{\lambda}\,x + b \sinh\sqrt{\lambda}\,x$ $\leftarrow$ $X(0) = 0$ yields $a = 0$.

or $= A e^{\sqrt{\lambda}\,x} + B e^{-\sqrt{\lambda}\,x}$

$X(\pi) = b \sinh\sqrt{\lambda}\,\pi = 0$

$\Rightarrow$ $b = 0$ too.

$X(x) \equiv 0$.

<u>Case $\lambda < 0$</u>: Write $\lambda = -k^2$.

$$X'' + k^2 X = 0$$

$$r^2 + k^2 = 0$$

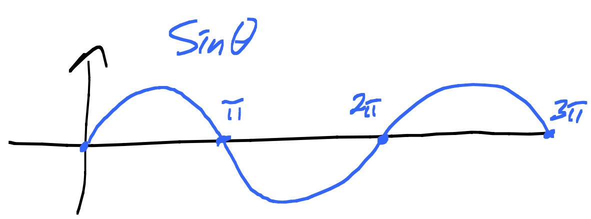
$$r = \pm ki$$

$$X(x) = \cancel{A \cos kx} + B \sin kx \qquad X(0) = A = 0.$$

$$X(\pi) = B \sin k\pi = 0$$

don't want $B = 0$

Need $\sin k\pi = 0$



Need $k\pi = n\pi$ $\quad n = 1, 2, 3, \ldots$

$$\boxed{k = n}$$

<u>For $k = n$</u>: $X_n(x) = 0 \cos nx + B \sin nx$

$$T_n(t) = C e^{\lambda t} = C e^{-n^2 t}$$

$$u_n(x,t) = X_n(x)T_n(t) = e^{-n^2 t} \sin nx$$

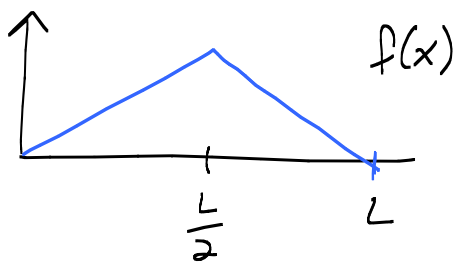
<u>Big idea</u>: Try $\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx}$

<u>Last thing</u>: I.C. $u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$ ← Fourier Sine series!

$\overset{\text{want}}{=} f(x)$

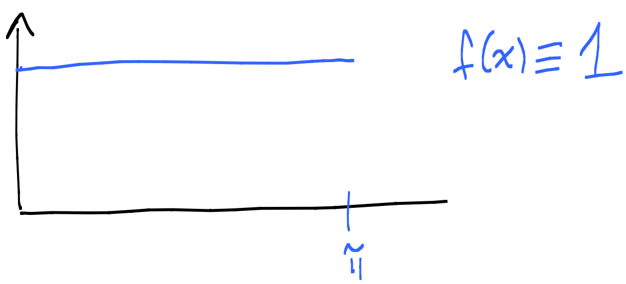
$$\boxed{B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx}$$

<u>Prob</u>:



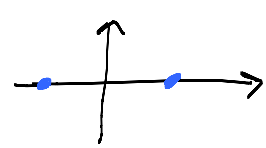
HWK 5 : prob 1

<u>Prob</u>:



$$B_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx \, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi n}\left[-\underbrace{\cos n\pi}_{(-1)^n} - (-1)\right]$$



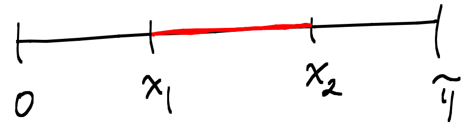
$$= \frac{2}{\pi n}\left(1 + (-1)^{n+1}\right)$$

$$= \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$u(x,t) = \frac{4}{\pi}\left(\frac{1}{1} e^{-1^2 \cdot t} \sin x + \frac{1}{3} e^{-3^2 t} \sin 3x + \frac{1}{5} e^{-5^2 t} \sin 5x + \cdots \right)$$

$$= \boxed{\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-(2n-1)^2 t} \sin (2n-1)x}$$

<u>Derive Heat eqn</u>:



Total heat content between x_1 and x_2:

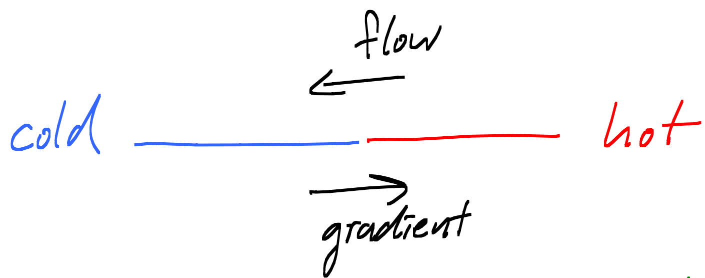
$$c \int_{x_1}^{x_2} u(x,t)\, dx$$

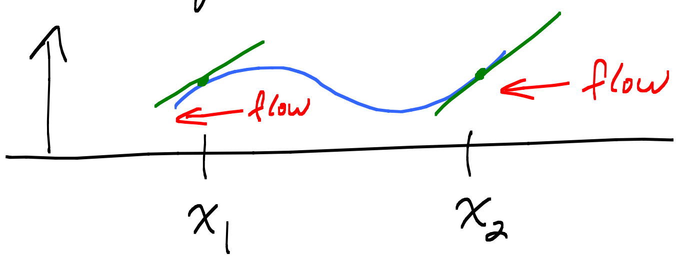
Rate it is changing: $\frac{\partial}{\partial t} \left(c \int_{x_1}^{x_2} u(x,t)\, dx \right)$

$$= \int_{x_1}^{x_2} c \frac{\partial u}{\partial t}(x,t)\, dx$$

Cause of change: Heat flow via Kelvin's law:

$$(\text{Rate of heat flow}) = -k\left(\underbrace{\text{temp gradient}}_{\frac{d}{dx}\text{temp}}\right)$$





Rate of change

$$\int_{x_1}^{x_2} c\,\frac{\partial u}{\partial t}(x,t)\,dx = k\,\frac{\partial u}{\partial x}(x_2,t) - k\,\frac{\partial u}{\partial x}(x_1,t)$$

$$= \int_{x_1}^{x_2} k\,\frac{\partial^2 u}{\partial x^2}(x,t)\,dx$$

$$\int_{x_1}^{x_2}\left[\underbrace{c\,\frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2}}_{\text{must be } \equiv 0\,!}\right]dx = 0$$