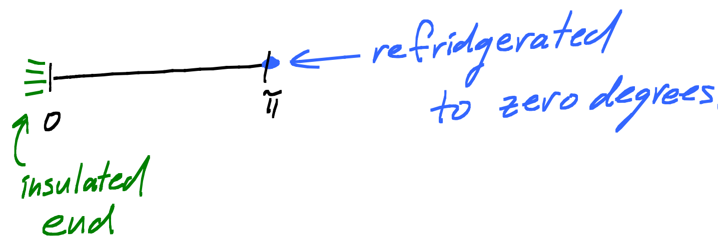
Lesson 20 More hot wire problems

Monday review

Sturm-Liouville problem from having



$$X'' - \lambda X = 0$$

B.C.'s ; $\begin{cases} X'(0) = 0 \\ X(\pi) = 0 \end{cases}$

Fact: Non-zero X's corresponding to different λ's are $\perp$ on $[0, \pi]$.

Terminology: $\underbrace{X'' = \lambda X}_{}$ Let $L = \dfrac{d^2}{dx^2}$

$LX = \lambda X$ $\leftarrow$ Think: X is a eigenfunction for L and λ is an eigenvalue.

Sturm-Liouville $LX_j = \lambda_j X_j$, $\lambda_1 \neq \lambda_2$. Then $X_1 \perp X_2$.

Why: $\lambda_1 \langle X_1, X_2 \rangle = \langle \underbrace{\lambda_1 X_1}_{LX_1}, X_2 \rangle$

$$= \int_0^{\pi} X_1'' X_2 \, dx = \int_0^{\pi} \underbrace{X_2}_{u} \underbrace{X_1'' \, dx}_{dv} \qquad v = X_1'$$

$$= \underbrace{X_2 X_1' \Big|_0^{\pi}}_{\text{0 because of B.C.'s!}} - \int_0^{\pi} X_1' \underbrace{X_2'}_{u} \, dx$$

$X_1' \, dx = dV$

$V = X_1$

$$= \underbrace{-X_2' X_1 \Big|_0^{\pi}}_{=0 \text{ by B.C.'s}} + \int_0^{\pi} X_1 X_2'' \, dx$$

$$= \langle X_1, \underbrace{X_2''}_{LX_2 = \lambda_2 X_2} \rangle = \langle X_1, X_2 \rangle \lambda_2$$

<u>Summary</u>: $\lambda_1 \langle X_1, X_2 \rangle = \langle X_1, X_2 \rangle \lambda_2$

$(\lambda_1 \neq \lambda_2)$ $\quad \underbrace{(\lambda_1 - \lambda_2)}_{\text{not zero}} \underbrace{\langle X_1, X_2 \rangle}_{\text{must be zero}} = 0$

Always get fcns X_n leading to Fourier series-like orthogonal expansions!

<u>Hot wire problem</u>: Radiaditive conditions at ends.

Rate of heat loss is proportional to temp.

$X'(x) = +k X(x)$

Rate of heat loss is also proportional to temp gradient.

$X'(x) = - k X(x)$

New kind of BC: $\begin{cases} X'(0) - k X(0) = 0 \\ X'(\pi) + k X(\pi) = 0 \end{cases}$

Sturm-Liouville theory shoes eigenfunctions are $\perp$!

Mixed problems : refridgerated, insulated, radiative

all mixed up at ends.

Euler equations : r equation in polar coordinates when you do separation of variables in a problem involving Δ.

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

Try

$$y = x^r$$
$$y' = rx^{r-1}$$
$$y'' = r(r-1)x^{r-2}$$

want

Plug into ODE $\quad ar(r-1)x^r + brx^r + cx^r = 0$

$$\Big[\underbrace{ar(r-1) + br + c}_{\text{need this } = 0}\Big]x^r = 0$$

Case roots r_1, r_2 real and $\neq$. Best, easiest case.

Gen^l sol^n $\quad y = c_1 x^{r_1} + c_2 x^{r_2}$

Note: x^{r_1}, x^{r_2} lin. independent ; $\quad \dfrac{x^{r_1}}{x^{r_2}} = x^{r_1 - r_2} \leftarrow$ not a const.

equiv to : Wronskian not zero.

Case: $r_1 = r_2$ (real). $\quad$ [Quad formula $r_1, r_2 = \dfrac{-(\ ) \pm \sqrt{\quad}}{2(\ )(\ )}$]

Only get one sol^n x^{r_1}.

$$L(y) = ax^2y'' + bxy' + cy$$

$$L[x^r] = \underbrace{[\text{ poly }]}_{a(r-r_1)^2} x^r \leftarrow$$

Euler: Take $\frac{\partial}{\partial r}$!

$$\frac{\partial}{\partial r} L[x^r] = L\Big[\underbrace{\tfrac{\partial}{\partial r} x^r}\Big] = a\,2\underbrace{(r-r_1)}_{\substack{\text{zero}\\ \text{if}\\ r=r_1}} x^r + a\underbrace{(r-r_1)}^2 \underbrace{\frac{\partial}{\partial r} x^r}_{x^r \ln x}$$

$$\frac{\partial}{\partial r} x^r = \frac{\partial}{\partial r} e^{r\ln x}$$

$$= \underbrace{e^{r\ln x}}_{x^r} \cdot \ln x$$

$$= x^r \ln x$$

Aha! Now let $r = r_1$. $L[x^{r_1}\ln x] = 0$.

Found second sol^n! $y_2 = x^{r_1}\ln x \leftarrow$ lin. indep from x^{r_1} (Quotient $= \ln x$, not a const.)

Gen^l Sol^n: $y = c_1 x^{r_1} + c_2 x^{r_1}\ln x$.

Case: Complex case $r = \alpha \pm \beta i$.

$$x^r = e^{r\ln x}$$

Euler: Consider $r = \alpha + \beta i$

$$x^{\alpha+\beta i} = e^{(\alpha+\beta i)\ln x} = \underbrace{e^{\alpha \ln x}}_{x^\alpha} \underbrace{e^{i\beta\ln x}}_{\cos(\beta \ln x) + i\sin(\beta\ln x)}$$

Complex sol^n: $\underbrace{x^\alpha \cos(\beta \ln x)}_{y_1 \text{ real sol}^n!} + i\, \underbrace{x^\alpha \sin(\beta \ln x)}_{y_2 \text{ another!}}$

Independent!

$$0 = L[\underbrace{u+iv}_{\text{complex sol}^n}] = \underbrace{L[u]}_{=0} + i\, \underbrace{L[v]}_{=0}$$

one complex yields two real sol^{n}s.

Gen^l Sol^n: $c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x)$