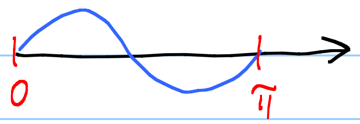
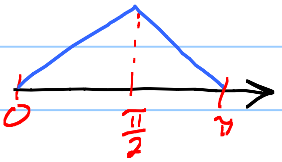
Lesson 2 Sol^n of the harpsichord problem



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

BC: $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

IC: $\begin{cases} u(x,0) = \\ \frac{\partial u}{\partial t}(x,0) \equiv 0 \leftarrow \text{pluck} \end{cases}$



Johann Bernoulli's sol^n: Try $u(x,t) = X(x)T(t)$

$$\boxed{\begin{gathered} X'' - \lambda X = 0 \\ \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \end{gathered}} \qquad T'' - \lambda T = 0$$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

ODE's!

Case $\lambda = 0$: $X \equiv 0$

Case $\lambda > 0$: $X \equiv 0$

Case $\lambda < 0$: Write $\lambda = -k^2$. $r^2 + k^2 = 0$ $r = \pm ki$

$$e^{ikx} = \underbrace{\cos kx} + i\,\underbrace{\sin kx} \text{ is a complex sol}^n$$

two real sol^{n}s

Linear, homog: $X = c_1 \cos kx + c_2 \sin kx$ are sol^{n}s,

It is the gen^l sol^n because we can solve any and all initial value problems with it: $\begin{cases} X(0) = A \\ X'(0) = B \end{cases}$

Key: Wronskian $$\det\begin{bmatrix}\cos kx & \sin kx\\ -k\sin kx & k\cos kx\end{bmatrix} = \underbrace{k\cdot 1}_{\neq 0}$$

Meaning of linear: PDE $$\underbrace{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}}_{Lu} = \underset{\nwarrow \text{ homog}}{0}$$

$$L[c_1u_1 + c_2u_2] = c_1Lu_1 + c_2Lu_2$$

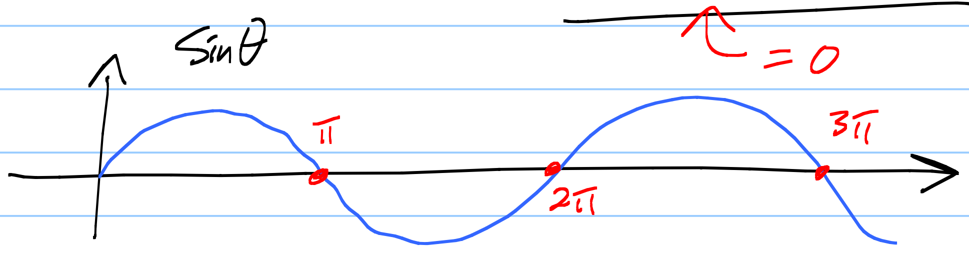
Back to $X(x) = \cancel{c_1\cos kx} + c_2\sin kx$

BC: $X(0) = c_1\cos 0 + c_2\sin 0 = c_1 = 0$

need

$X(\pi) = 0\cdot\cos k\pi + c_2\sin k\pi = 0$

Need $\boxed{c_2\underbrace{\sin k\pi}_{\nwarrow = 0} = 0}$ Don't want $c_2=0$ too!



Need $k\pi = n\pi$ for some $n\in\mathbb{N}$.

$\boxed{k=n}$ $n=1,2,3,\dots$

$X_n(x) = \sin nx$ (Take $c_2=1$).

Next: $T'' - \lambda T = 0 \qquad \lambda = -k^2 = -n^2.$

$$T_n(t) = A_n \cos nt + B_n \sin nt$$

$u_n(x,t) = X_n(x) \cdot T_n(t)$ are infinitely many sol^{n}s.

Superposition princ. (linear): So are linear combo's!

[BC are homog too!]

$$u(x,t) = \sum_{n=1}^{N} \sin nx \left(A_n \cos nt + B_n \sin nt \right)$$

Finally: IC: Ouch! Not enough sol^{n}s.

Big idea let $N \to \infty$.

$$\text{IC: } \begin{cases} u(x,0) = \displaystyle\sum_{n=1}^{\infty} A_n \sin nx \overset{\text{want}}{=} f(x) \leftarrow \text{pluck shape} \\ \dfrac{\partial u}{\partial t}(x,0) = \displaystyle\sum_{n=1}^{\infty} \sin nx \left(-n A_n \cdot 0 + n B_n \cdot 1 \right) = 0 \end{cases}$$

Aha! Take all B_n's $= 0$!

Boils down to: Need $\boxed{\displaystyle\sum_{n=1}^{\infty} A_n \sin nx = f(x)}$

Fourier sine series

Key to Fourier series:

$\{\sin nx\}_{n=1}^{\infty}$ are "orthogonal" on $[0,\pi]$.

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$n=m$ case : $\displaystyle\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1}{2}(1-\cos 2nx)\, dx = \frac{\pi}{2}$

Idea: $\left[\vec{V} = c_1 \vec{u}_1 + \cdots + c_m \vec{u}_m + \cdots + c_N \vec{u}_N\right] \cdot \vec{u}_m$

(orthog vect: $\vec{u}$'s)

$$= c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_m}_{0} + \cdots + c_m \underbrace{\vec{u}_m \cdot \vec{u}_m}_{\neq 0} + \cdots + c_N \underbrace{\vec{u}_N \cdot \vec{u}_m}_{0}$$

$$c_m = \frac{\vec{V} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m}$$

$$f(x) \cdot \sin mx = \left(\sum_{n=1}^{\infty} A_n \sin nx \right) \sin mx$$

$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} \left(A_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\substack{\text{zero if } n \neq m \\ \frac{\pi}{2} \text{ if } n = m}} \right)$$

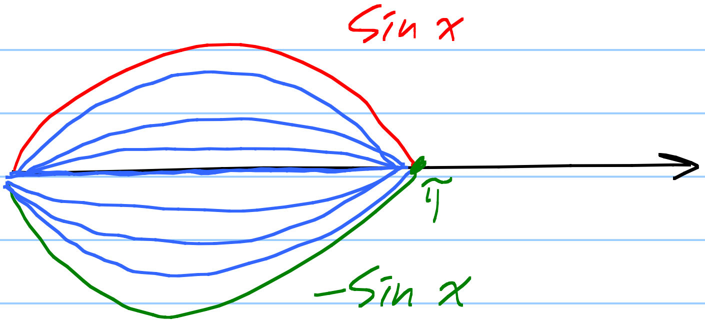
$$= A_m \cdot \frac{\pi}{2}$$

Must have $$\boxed{A_m = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin mx\,dx}$$

can change m to n.

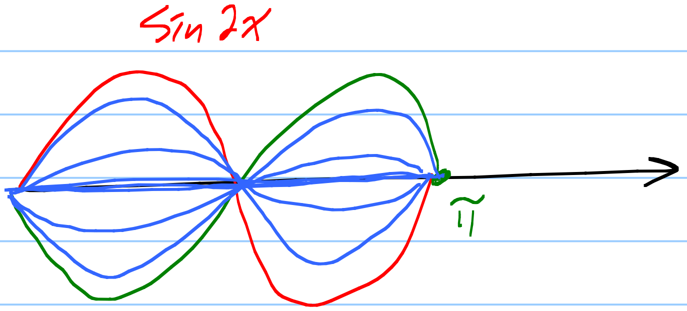
Modes of oscillation: $u_n(x,t) = \sin nx \cos nt$

$n=1$:



The "pitch" of the sound

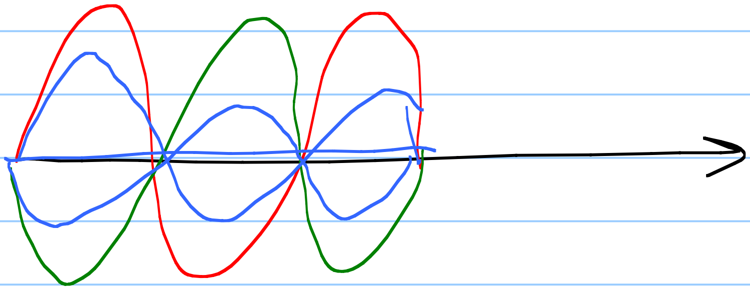
$n=2$:



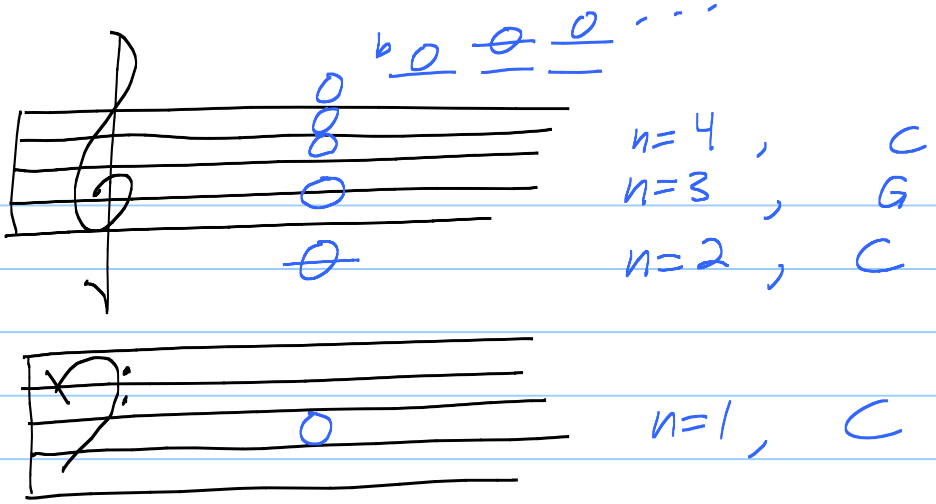
Touch a guitar string in middle.

Pitch jumps an octave

$n=3$:



$\sin 3x \cos 3t$


n=4 , C
n=3 , G
n=2 , C
n=1 , C