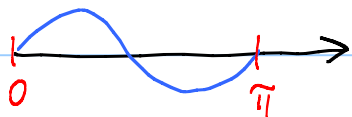


Lesson 2 Solⁿ of the harpsichord problem



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$BC: \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

$$IC: \begin{cases} u(x,0) = \text{triangle wave} \\ \frac{\partial u}{\partial t}(x,0) = 0 \leftarrow \text{pluck} \end{cases}$$

Johann Bernoulli's solⁿ: Try $u(x,t) = X(x)T(t)$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$$T'' - \lambda T = 0$$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

ODE's!

Case $\lambda = 0$: $X \equiv 0$

Case $\lambda > 0$: $X \equiv 0$

Case $\lambda < 0$: Write $\lambda = -k^2$. $r^2 + k^2 = 0$ $r = \pm ki$

$$e^{ikx} = \underbrace{\cos kx}_{\uparrow} + i \underbrace{\sin kx}_{\uparrow} \text{ is a complex sol}^n$$

two real solⁿs

Linear, homog : $X = c_1 \cos kx + c_2 \sin kx$ are solⁿs,

It is the gen^l solⁿ because we can solve any and all initial value problems with it : $\begin{cases} X(0) = A \\ X'(0) = B \end{cases}$

Key: Wronskian

$$\det \begin{bmatrix} \cos kx & \sin kx \\ -k \sin kx & k \cos kx \end{bmatrix} = \underbrace{k \cdot 1}_{\neq 0}$$

Meaning of linear: PDE $\underbrace{\frac{\partial^2 u}{\partial x^2}}_{Lu} - \frac{\partial^2 u}{\partial x^2} = 0$ $\nearrow$ homog

$$L[c_1 u_1 + c_2 u_2] = c_1 L u_1 + c_2 L u_2$$

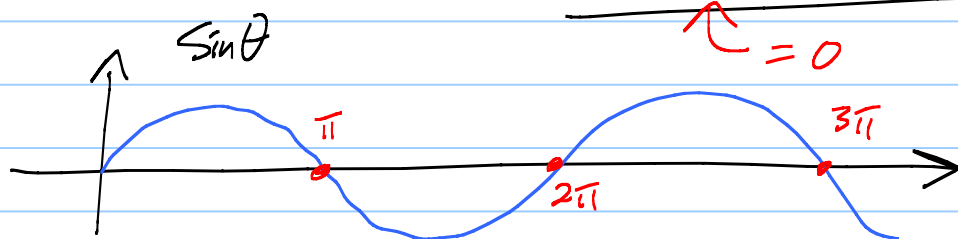
Back to $X(x) = c_1 \cos kx + c_2 \sin kx$

BC: $X(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$

$X(\pi) = 0 \cdot \cos k\pi + c_2 \sin k\pi = 0$

Need $\boxed{c_2 \sin k\pi = 0}$

Don't want $c_2 = 0$ too!



Need $k\pi = n\pi$ for some $n \in \mathbb{N}$.

$$\boxed{k = n}$$

$n = 1, 2, 3, \dots$

$X_n(x) = \sin nx$ (Take $c_2 = 1$).

Next: $T'' - \lambda T = 0$ $\lambda = -k^2 = -n^2$.

$$T_n(t) = A_n \cos nt + B_n \sin nt$$

$u_n(x, t) = X_n(x) T_n(t)$ are infinitely many solⁿs.

Superposition princ. (linear): So are linear combo's!

[BC are homog too!]

$$u(x, t) = \sum_{n=1}^N \sin nx (A_n \cos nt + B_n \sin nt)$$

Finally: IC: Ouch! Not enough solⁿs.

Big idea let $N \rightarrow \infty$.

IC: $u(x, 0) = \sum_{n=1}^{\infty} A_n \sin nx = f(x)$ want ← pluck shape

$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \sin nx (-n A_n \cdot 0 + n B_n \cdot 1) = 0$

Aha! Take all B_n 's = 0!

Boils down to: Need

$$\sum_{n=1}^{\infty} A_n \sin nx = f(x)$$

Fourier sine series

Key to Fourier series :

$\{\sin nx\}_{n=1}^{\infty}$ are "orthogonal" on $[0, \pi]$. 4

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$$n=m \text{ case: } \int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \frac{1}{2} (1 - \cos 2nx) \, dx = \frac{\pi}{2}$$

Idea: $\left[\vec{v} = c_1 \vec{u}_1 + \dots + c_m \vec{u}_m + \dots + c_N \vec{u}_N \right] \cdot \vec{u}_m$
 (orthog vect: $\vec{u}$'s)

$$= c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_m}_0 + \dots + c_m \underbrace{\vec{u}_m \cdot \vec{u}_m}_{\neq 0} + \dots + c_N \underbrace{\vec{u}_N \cdot \vec{u}_m}_0$$

$$c_m = \frac{\vec{v} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m}$$

$$f(x) \cdot \sin mx = \left(\sum_{n=1}^{\infty} A_n \sin nx \right) \sin mx$$

$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} \left(A_n \int_0^{\pi} \sin nx \sin mx \, dx \right)$$

zero if $n \neq m$
 $\frac{\pi}{2}$ if $n = m$

$$= A_m \cdot \frac{\pi}{2}$$

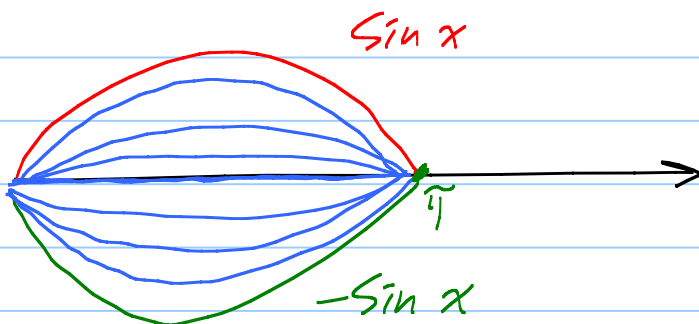
Must have

$$A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx$$

↑ can change m to n .

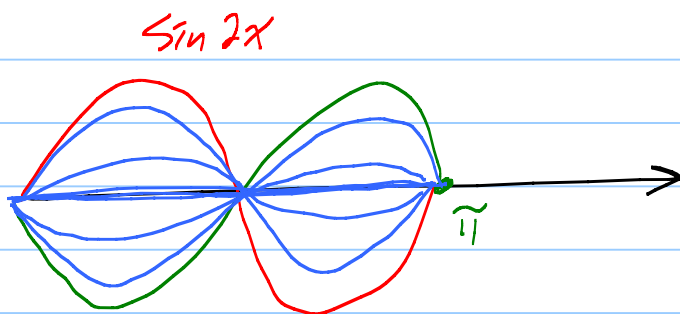
Modes of oscillation: $u_n(x, t) = \sin nx \cos nt$

$n=1$:



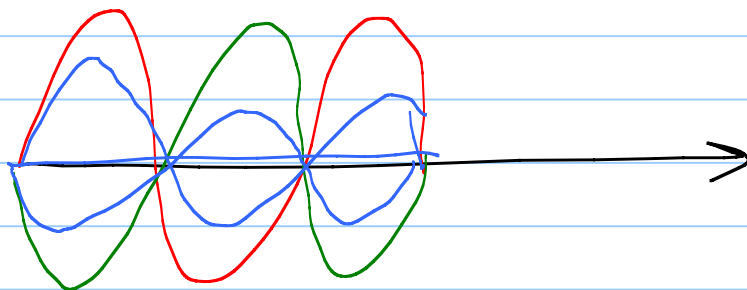
The "pitch"
of the sound

$n=2$:



Touch a guitar
string in middle.
Pitch jumps an
octave

$n=3$:



$\sin 3x \cos 3t$

Handwritten musical notation on a treble clef staff. The staff contains three blue circles on the first three lines (F, C, G) and a blue circle on the first space (D). Above the staff, there are blue circles on the first three lines (F, C, G) and a blue circle on the first space (D), with a blue 'b' above the first circle and a blue '...' to the right. To the right of the staff, the following text is written:

$n=4$, C
 $n=3$, G
 $n=2$, C

Handwritten musical notation on a bass clef staff. The staff contains a blue circle on the second line (G). To the right of the staff, the following text is written:

$n=1$, C