Lesson 6 Complex exponential

HWK 1 due tonight 11:00 pm in Gradescope

$$f(x) \cong a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right) \overset{!}{=} \sum_{n=-N}^{N} c_n e^{inx}$$

$$\boxed{\begin{array}{c} c_0 = a_0, \quad c_n = \frac{1}{2}(a_n - i b_n) \quad n = 1, 2, 3, \ldots \\ c_{-n} = \frac{1}{2}(a_n + i b_n) \end{array}}$$

$$\begin{cases} a_n = c_n + c_{-n} \\ i b_n = -c_n + c_{-n} \end{cases} \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Def^n: $\sum_{n=-N}^{N} c_n e^{inx}$ is called a trigonometric polynomial

Complex exponential $z = x + iy$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Today only: Write $E(x+iy) = (e^x \cos y) + i(e^x \sin y)$

Properties: 0) $E(0) = E(0+i0) = \underline{1}$

1) $E(z+w) = E(z)E(w)$ (Group homomorphism)

2) $E(z-z) = E(0) = 1 = E(z)E(-z)$

Consequently: $E(z) \neq 0$ for all z!

$$E(-z) = 1/E(z)$$

3) $E(z-w) = E(z)/E(w)$

4) $E(x) = E(x+i0) = e^x$ ← $E(z)$ extends e^x from $\mathbb{R}$ to $\mathbb{C}$

5) $E(iy) = E(0+iy) = \cos y + i \sin y$

Proof of (1): $E(z_1)E(z_2) = E(x_1+iy_1)\,E(x_2+iy_2)$

$$= e^{x_1}(\cos y_1 + i\sin y_1)\cdot e^{x_2}(\cos y_2 + i \sin y_2)$$

$$= \underbrace{e^{x_1} e^{x_2}}_{e^{x_1+x_2}}\Big[\underbrace{(\cos y_1 \cos y_2 \overset{i^2}{-} \sin y_1 \sin y_2)}_{\cos(y_1+y_2)} + i\underbrace{(\cos y_1 \sin y_2 + \cos y_2 \sin y_1)}_{\sin(y_1+y_2)}\Big]$$

$$= E\big((x_1+x_2)+i(y_1+y_2)\big) = E(z_1+z_2) \checkmark$$

Simple trick: Get addition formulas from

$$e^{iy_1}e^{iy_2} = e^{i(y_1+y_2)}$$

DeMoivre's formula: $\left(e^{i\theta}\right)^N = e^{iN\theta} = \cos N\theta + i \sin N\theta$

Tool for finding N-th roots of complex #s.

Pf: $(e^{i\theta})^N = \underbrace{e^{i\theta}\cdot e^{i\theta}\cdots e^{i\theta}}_{N \text{ times}} = e^{iN\theta}$

Remark: DeMoivre's true if $N \in \mathbb{Z}$.

Fun fact $E(z)$ is complex diff'ble

$$\lim_{z \to a} \frac{E(z)-E(a)}{z-a} \text{ exists and}$$

$$\boxed{E'(z) = E(z)}$$

Add condition: $E(0) = 1$

uniquely determine complex exponential

Fact: $E(z)$ is the unique $\mathbb{C}$-diff'ble fcn on $\mathbb{C}$ extending e^x from $\mathbb{R}$ to $\mathbb{C}$.

EX: D.Q. for $\mathbb{C}$-derivative of z^2:

$$\frac{z^2-a^2}{z-a} = \frac{(\cancel{z-a})(z+a)}{\cancel{z-a}} = z+a \to a+a = 2a$$

as $z \to a$.

Important function: Dirichlet kernel

$$D_N(t) = \sum_{n=-N}^{N} e^{int} = \sum_{n=1}^{N} e^{-int} + \sum_{n=0}^{N} e^{int}$$

$$= \sum_{n=1}^{N} (\cos nt - i \sin nt) + \sum_{n=0}^{N} (\cos nt + i \sin nt)$$

$$= \underset{\substack{\uparrow \\ n=0}}{1} + 2\sum_{n=1}^{N} \cos nt = \frac{\sin\left[\left(N+\frac{1}{2}\right)t\right]}{\sin \frac{t}{2}}$$

Why: $1 + \cos t + \cos 2t + \cdots + \cos Nt$

$$= \mathrm{Re}\left[1 + e^{it} + e^{i2t} + \cdots + e^{iNt}\right]$$

$$= \mathrm{Re}\left[1 + (e^{it}) + (e^{it})^2 + \cdots + (e^{it})^N\right] \quad *$$

Aha! Geometric zeries! $z = e^{it}$

$$S_N = 1 + z + z^2 + \cdots + z^N$$

$$- \quad zS_N = \quad z + z^2 + \cdots + z^N + z^{N+1}$$

$$(1-z)S_N = 1 - z^{N+1}$$

$$S_N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z} \to \frac{1}{1-z}$$

if $|z| < 1$

$(*)$: $$\mathrm{Re}\left[\frac{1 - (e^{it})^{N+1}}{1 - (e^{it})}\right]$$

$$= \mathrm{Re}\left[\frac{1 - e^{i(N+1)t}}{1 - e^{it}} \cdot \frac{e^{-it/2}}{e^{-it/2}} \cdot \frac{\left(\frac{-1}{2i}\right)}{\left(\frac{-1}{2i}\right)}\right]$$

$$= \mathrm{Re}\left[\frac{\frac{i}{2}e^{-it/2} - \frac{i}{2}e^{i(N+\frac{1}{2})t}}{\left(\frac{e^{it/2} - e^{-it/2}}{2i}\right)}\right] \leftarrow \sin\frac{t}{2}$$

$$= \frac{\frac{1}{2}\sin\frac{t}{2} + \frac{1}{2}\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\sin(N+\frac{1}{2})t}{\sin t/2} = \underbrace{1 + \cos t + \cdots + \cos Nt}_{\text{Red}}$$

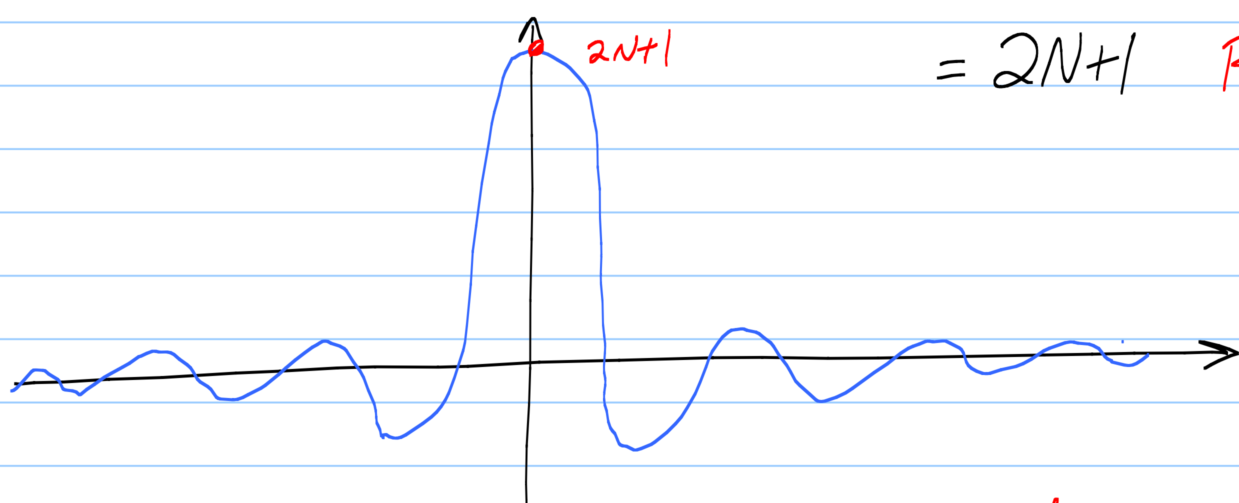
Finally, $$D_N(t) = 1 + 2\sum_{n=1}^{N} \cos nt$$

$$= 2(\text{Red}) - 1$$

$$= \frac{\sin(N+\frac{1}{2})t}{\sin t/2}$$

L'Hôpital's at $t=0$: $$\lim_{t\to 0} D_N(t) = \frac{(N+\frac{1}{2})\cdot\cos 0}{\frac{1}{2}\cdot\cos 0}$$

$$= 2N+1 \quad \text{Big!}$$



Key property : $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1 \quad \text{(easy)}$$

$$\left(\int_{-\pi}^{\pi} \cos nt = 0\right)$$