Lesson 9 Convergence of Fourier series

HWK 2 due in GS by 11:00pm Wed, 2/1

$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} \underset{c_n = \hat{f}(n)}{c_n} e^{inx}$$

Stein

$$= \int_{-\pi}^{\pi} D_N(x-t) f(t)\, dt = \int_{-\pi}^{\pi} D_N(t) f(x-t)\, dt$$

extend f from $[-\pi, \pi]$ to be 2π-periodic on $\mathbb{R}$

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} \underbrace{D_N(t)}_{\frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})t}{\sin t/2}} \left[f(x-t) - f(x) \right] dt$$

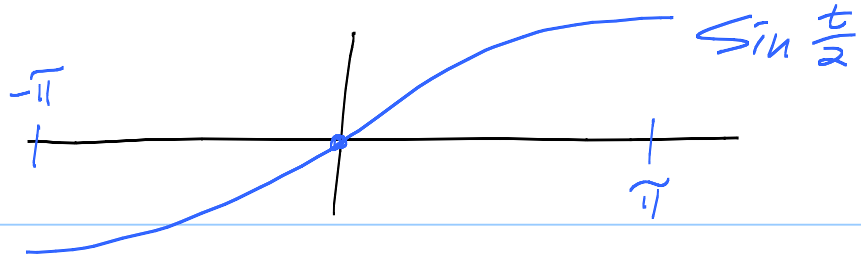
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(N+\tfrac{1}{2})t \underbrace{\left[\frac{f(x-t) - f(x)}{\sin t/2} \right]}_{g(t)} dt$$

If f is "nice", so is g.

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = t \underbrace{\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\right)}_{H(t)}$$

Ratio test shows R. of C. for series for H is ∞.

So H is C^∞-smooth on $\mathbb{R}$! $H(0) = 1$.



$H(t)$ is never zero.

Trick: $f(x+\Delta x) - f(x) = f(x+\tau\Delta x)\Big|_0^1$

$$= \int_0^1 \frac{\partial}{\partial\tau} f(x+\tau\Delta x)\, d\tau$$

$$= \int_0^1 \Delta x\, f'(x+\tau\Delta x)\, d\tau$$

$$= \Delta x \underbrace{\int_0^1 f'(x+\tau\Delta x)\, d\tau}$$

Nice! If f is C^k-smooth, this is C^{k-1}-smooth.

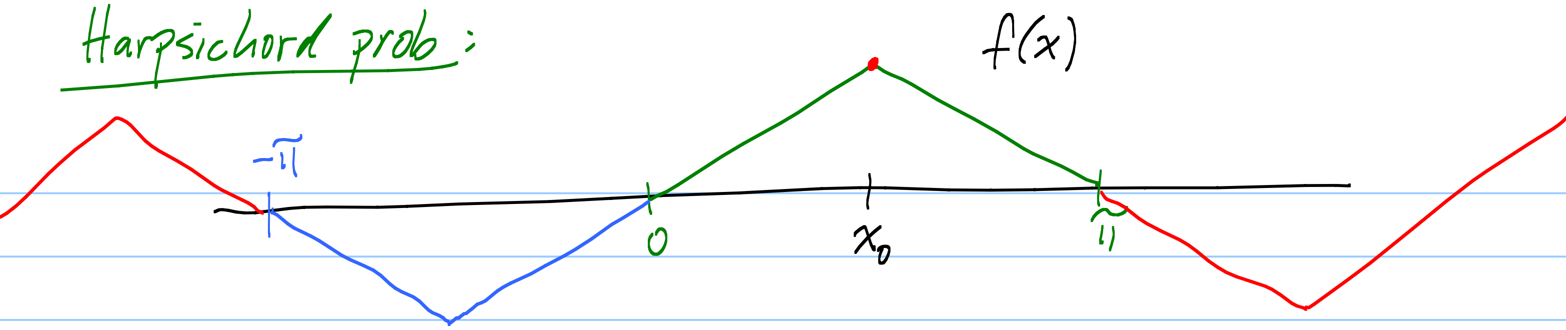
Back to $f(x-t) - f(x) = f(x+\underbrace{(-t)}_{\Delta x}) - f(x)$

$$= (-t)\underbrace{\int_0^1 f'(x+\tau(-t))\, d\tau} \leftarrow \int = H(x,t)$$

If f is C^k-smooth, this is C^{k-1}-smooth in x and t!

So $$g(t) = \frac{f(x-t)-f(x)}{\sin\frac{t}{2}} = \frac{(-t)\, H(x,t)}{t\cdot h(t)}$$

Harpsichord prob:



Aha! $f(x)$ satisfies a Lipschitz condition at x_0,

meaning that $|f(x_0) - f(x)| \le C\,|x_0 - x|$

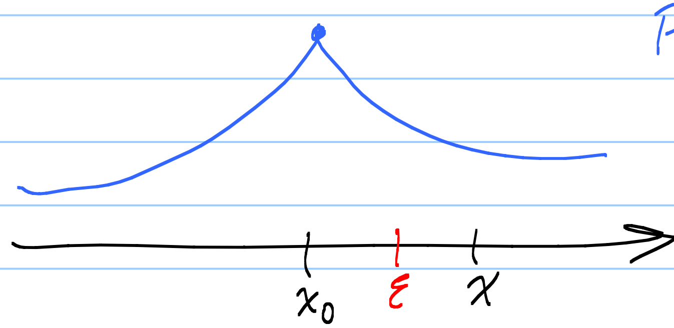
$$|f(x_0 - t) - f(x_0)| \le C\,|t|$$

Aha! $g(t)$ bounded near x_0 when have Lip condition,

Get convergence of Fourier series at kinks!

Lip estimate at

Piecewise C' kink



Mean value theorem $\dfrac{f(x) - f(x_0)}{x - x_0} = f'(\xi)$

$$|f(x) - f(x_0)| \le C\,|x - x_0| \quad \text{where } C = \max_{[-\pi,\pi]} |f'|.$$

Feeling: The smoother a fcn is, the faster the F.S. converges.

Why: Integration by parts:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Bessel's $\Rightarrow c_n \to 0$

$$f'(x) \overset{?}{\sim} \sum_{n=-\infty}^{\infty} in\, c_n e^{inx}$$

Yes!

$$f''(x) \overset{?}{\sim} \sum_{n=-\infty}^{\infty} \underbrace{(in)^2}_{-n^2} c_n e^{inx}$$

Bessel's $\Rightarrow n^2 c_n \to 0$!

$$\boxed{c_n = \hat{f}(n)}$$

Pf that $\hat{f'}(n) = in\,\hat{f}(n)$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{u}\, \underbrace{e^{-inx}\, dx}_{dv}$$

$dv = (\cos nx - i \sin nx)\, dx$

$du = f'(x)\, dx \qquad v = \int e^{-inx}\, dx = \frac{-1}{in} e^{-inx}$

$$= \frac{1}{2\pi}\left[uv\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v\, du \right]$$

$$= \frac{1}{2\pi}\left[\underbrace{f(x)\frac{-1}{in} e^{-inx}\Big|_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \frac{-1}{in} e^{-inx} f'(x)\, dx \right]$$

2π-periodic!

$$= \frac{1}{in}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx \right]$$

$[\ ] = \hat{f'}(n)$

Get $\widehat{f'}(x) = in\,\hat{f}(n)$ ✓

Can repeat!

Consequence: f C^2-smooth, 2π-periodic.

Bessel's: $c_n = \hat{f}(n) \to 0$ as $n \to \infty$.

[So $\exists\ M > 0$ such that $|c_n| < M$ for all n.]

Bessel's: $\widehat{f''}(n) = -n^2 \hat{f}(n) = -n^2 c_n \to 0$ too

So $\exists\ N$ such that $|-n^2 c_n| < 1$ if $n > N$

So $|c_n| < \frac{1}{n^2}$ when $n > N$.

c_n compare to tail end of convergent p-series

$\frac{\pi}{6} = \sum_{1}^{\infty} \frac{1}{n^2}$ $(p=2)$!

Weierstraß M-test $\Rightarrow$ F.S. for f converges uniformly to a continuous something. Today, we know that something is f!

Hmmm. Harpsichord prob. $f(x)$ continuous and piecewise C^1. Satisfies Lip cond at kink. Know F.S. converges at each pt to f.

Aha! $f(x) = \frac{1}{1^2} \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \cdots$

We have conditions of W. M-test because

$$\sum_{n \text{ odd}} \frac{1}{n^2} < \sum_{1}^{\infty} \frac{1}{n^2} < \infty.$$

Conclude: F.S. converges uniformly to f.