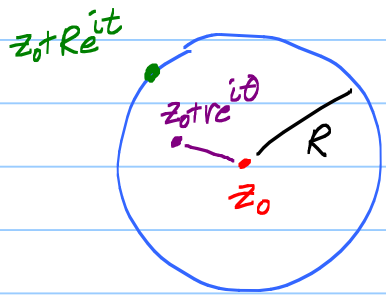
Lesson 15 Harmonic functions and the averaging property

HWK 3 due tonight 11:00 pm

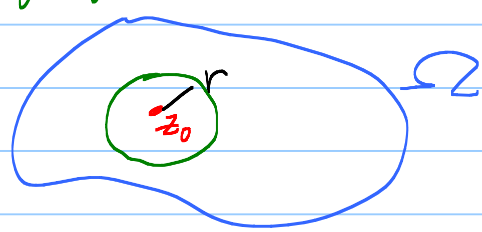
Poisson formula on a general disc $D_R(z_0)$:



$$u(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-t)}\, u(z_0+Re^{it})\,dt$$

Theorem: A continuous function that satisfies the averaging property is harmonic!

Ave prop



$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt$$

Pf: Ave property $\Rightarrow$ max principle

$$\max_{\overline{D_R(z_0)}} u = \max_{C_R(z_0)} u$$

$\leftarrow$ circle (boundary)

Given cont u on an open set Ω that satisfies ave prop.

Restrict attention to a disc $\overline{D_R(z_0)} \subset \Omega$ $\leftarrow$ open set

Clever idea: Define

U = Poisson integral of the boundary values of u on $C_R(z_0)$.

U solves the Dirichlet problem:

1) U cont on $\overline{D_R(z_0)}$
2) $\Delta U \equiv 0$ in $D_R(z_0)$
3) $U = u$ on $C_R(z_0)$

Aha! $u - U$ also satisfies the ave prop, and hence the max princ too. So

$$\max_{\overline{D_R(z_0)}} u - U = \max_{C_R(z_0)} \underbrace{u - U}_{\equiv 0} = 0$$

So $u - U \le 0$ on $\overline{D_R(z_0)}$.

Repeat argument using $U - u$ in place of $u - U$.

Get $U - u \le 0$ on $\overline{D_R(z_0)}$ too!

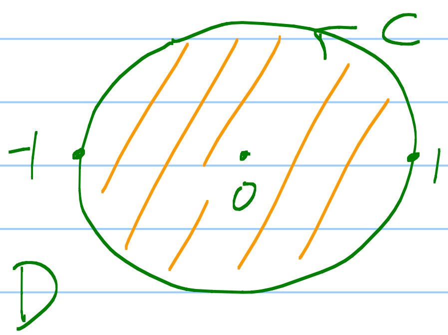
So $u \equiv U$ on $\overline{D_R(z_0)}$. So u is harm.

harmonic!

[Remark: u satisfies ave prop $\Longrightarrow$ $-u$ does too.]

Importance of Green's theorem and Green's identities.

Green's theorem easy and elem on a disc!

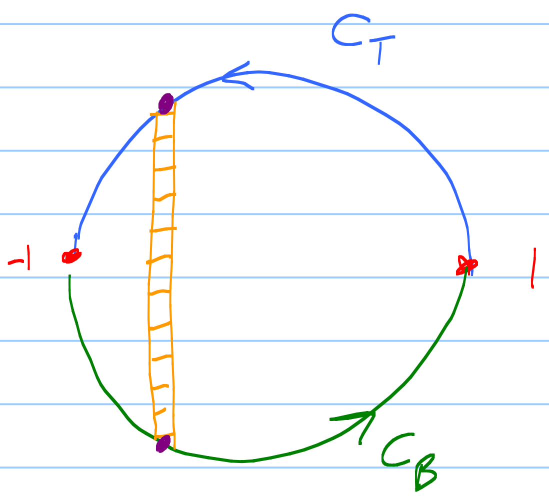


$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Parametrize C by curve $(x(t), y(t))$ as $a \le t \le b$.

$$\int_C P\,dx + Q\,dy = \int_a^b P(x(t), y(t))\,\underbrace{\dot{x}(t)\,dt}_{dx} \qquad dx = \frac{dx}{dt}\,dt$$

$$+ \int_a^b Q(x(t), y(t))\,\underbrace{\dot{y}(t)\,dt}_{dy}$$



C_B: $x(t) = t \qquad -1 \le t \le 1$

$y(t) = -\sqrt{1-t^2} = h_B(t)$

$-C_T$: $x(t) \qquad -1 \le t \le 1$

$y(t) = \sqrt{1-t^2} = h_T(t)$

<u>Claim</u>: $\displaystyle \int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dx\,dy$

<u>Pf</u>: $\displaystyle \int_C = \int_{C_B} + \int_{C_T} = \left(\int_{C_B} - \int_{-C_T}\right) P\,dx$

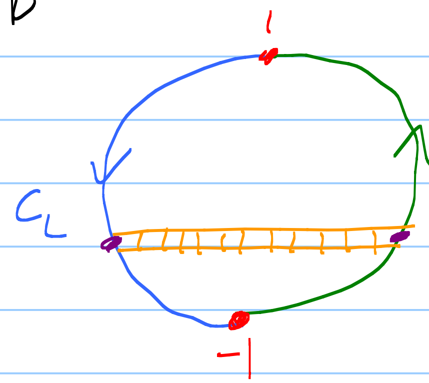
$$= \int_{-1}^{-1} P(t, h_B(t))\,\underset{\substack{\uparrow \\ \dot{x}(t)=1}}{1}\cdot dt - \int_{-1}^{1} P(t, h_T(t))\,1\cdot dt$$

$$= - \int_{-1}^{1} \left[P(t, h_T(t)) - P(t, h_B(t))\right] dt \leftarrow \text{change } t \text{ to } x \text{ now}$$

$$\int_{h_B(t)}^{h_T(t)} \frac{\partial P}{\partial y}(t,y)\,dy \leftarrow \text{fund thm of calculus}$$

$$= -\iint_D \frac{\partial P}{\partial y}\,dx\,dy$$

The other half



C_R : $y(t) = t$, $-1 \le t \le 1$

$x(t) = \sqrt{1-t^2} = g_R(t)$

$-C_L$: $y(t) = t$ $\quad -1 \le t \le 1$

$x(t) = -\sqrt{1-t^2} = g_L(t)$

$$\int_C Q\,dy = \left(\int_{C_R} - \int_{-C_L}\right) Q\,dy$$

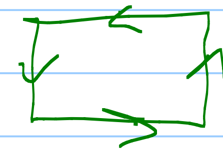
$$= \int_{-1}^{1} Q(g_R(t),t)\,\underset{\uparrow\ \dot{y}=1}{1}\cdot dt - \int_{-1}^{1} Q(g_L(t)\ t)\,dt$$

Change t to y here

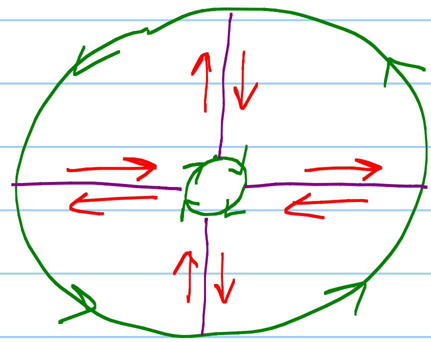
$$= \int_{-1}^{1}\left(\int_{g_L(y)}^{g_R(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\right)dy$$

$$= \iint_D \frac{\partial Q}{\partial x}\,dx\,dy \quad \checkmark$$

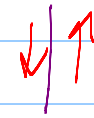
Can do same thing on any region that has a "top curve and a bottom curve" and a "left curve and a right curve".   Ex:



Cool thing:

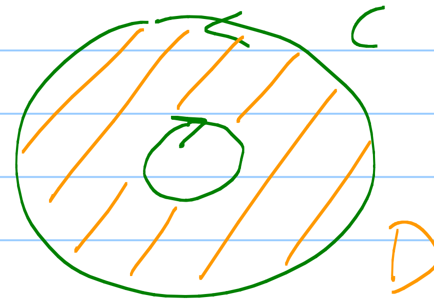


Write out Green's for four pieces of pie. Integrals



cancel

Green's theorem for an annulus:



C = (counterclockwise C_{outer}) ∪ (clockwise C_{inner})

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Green's theorem ⟹ $\underbrace{\text{Divergence theorem}}$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \underbrace{\operatorname{div}\vec{F}}_{=\nabla\cdot\vec{F}}\,dx\,dy$$

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} \qquad \operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$